Why QFT?

In quantum mechanics class, you've worked primarily with non-relativistic objects: the Hamiltonians you write down that are of the form:

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}) \tag{1}$$

use the non-relativistic dispersion relation $E = \boldsymbol{p}^2/2m$. If we wanted to include relativity, we would need to use the full dispersion $E^2 = (c\boldsymbol{p})^2 + (mc^2)^2$. We'll talk more about this later in the course, but it's good to remember that most of the QM you've studied previously is non-relativistic. Let's use the uncertainty relations we know from non-relativistic quantum mechanics and combine them with relativity to get a heuristic feel for why QFT is weird and when it comes into play:

- $\Delta x \cdot \Delta p \gtrsim \hbar$ (non-relativistic QM is not valid at short-distances). Consider an electron of mass m in a box of length L. QM works well to describe the physics of the electron when it is not relativistic, but if the box becomes too small then the electron must become relativistic. We can see from the uncertainty relation: the smaller the box is, the smaller Δx is, and so the larger the uncertainty Δp is to compensate. When Δp is on the order of mc, the electron is relativistic, i.e. when the box is of size:
$$L \sim \frac{\hbar}{mc} \equiv \lambda_c, \tag{2}$$
the system must be treated relativistically. This length scale is called the **Compton wavelength**.
- $\Delta E \cdot \Delta t \gtrsim \hbar$ (QM + relativity implies virtual particles). Consider probing a section of the vacuum somewhere in space for some characteristic timescale τ. As you make τ smaller and look at the vacuum for shorter and shorter periods of time, this uncertainty relation implies that ΔE gets larger and larger– there are large energy fluctuations in the area that you're probing. Now, relativity factors into this because it tells us that *energy and mass are the same*; as such, these large energy fluctuations can actually be interpreted as massive particles, with $\Delta E \sim Mc^2$. These particles only exist for a very short amount of time, and are constantly popping into and out of existence; the vacuum in QFT is not static and boring, but rather is dynamic and has its own interesting physics.

Heisenberg and Schrödinger

- The Heisenberg and Schrödinger pictures are two complementary views of looking at QM.
  - **Schrödinger picture**: states $|\psi(t)\rangle_S$ evolve with time and the operators $\mathcal{O}_S$ stay constant (up to explicit time dependence).
  - **Heisenberg picture**: states $|\psi\rangle_H$ stay constant, but the operators evolve with time, $\mathcal{O}_H(t)$.
- The correspondence between these two occurs because the physics must be the same, regardless of how we factor the time dependence into the operator or the states. We set both pictures equal at some reference time, which we conventionally take to be $t = 0$ to simplify things. This yields:

$$|\psi(t=0)\rangle_S = |\psi\rangle_H \qquad\qquad \mathcal{O}_S = \mathcal{O}_H(t=0). \tag{3}$$

- Time evolution: The time evolution operator in QM is given by a unitary operator $U(t)$. When the Hamiltonian is time-independent, the Schrödinger equation yields $U(t) = \exp(-iHt)$. States in the Schrödinger picture transform by the action of $U(t)$:
$$|\psi(t)\rangle_S = U(t)|\psi(t=0)\rangle_S. \tag{4}$$
Guaranteeing that the physics is equal implies that the expectation value of an operator in any state must be equal between the Schrödinger and Heisenberg pictures, $\langle\psi(t)|\mathcal{O}|\psi(t)\rangle_S = \langle\psi|\mathcal{O}(t)|\psi\rangle_H$, which immediately shows us how operators in the Heisenberg picture evolve with time:
$$\mathcal{O}_H(t) = U^\dagger(t)\mathcal{O}_H(0)U(t). \tag{5}$$

- In the Heisenberg picture, we can derive an evolution equation for operators by differentiating Eq. (5). This yields the **Heisenberg equations of motion** for the operator $\mathcal{O}_H$ (valid in field theory as well):
$$\frac{d\mathcal{O}_H}{dt} = -i[\mathcal{O}_H, H] + \underbrace{\frac{\partial O_H}{\partial t}}_{\text{explicit}}. \tag{6}$$

Classical vs quantum mechanics

- Hamiltonian mechanics looks surprisingly like quantum mechanics in the Heisenberg picture! For coordinates q_i, the Lagrangian is $L(q, \dot{q})$. The **momenta conjugate to** q_i and the **Hamiltonian** are:
$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \qquad\qquad H = p_i\dot{q}_i - L. \tag{7}$$
An **classical observable** is a function $A(q_i, p_j, t)$.

- **Poisson brackets** are the classical analog of commutators. For two observables A, B, we define:
$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{8}$$
Note that q_i and p_j satisfy the following relations:
$$\{q_i, p_j\} = \delta_{ij} \qquad\qquad \{q_i, q_j\} = 0 = \{p_i, p_j\} \tag{9}$$

- Time evolution: the Poisson bracket is useful because it **generates time translation** in classical mechanics. For an observable $A(q_i, p_i, t)$, Hamilton's equations and the chain rule imply:
$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial t} = \{A, H\} + \frac{\partial A}{\partial t} \tag{10}$$

- The **main idea behind quantization** is to promote the coordinates q_i and p_j into operators which satisfy the canonical commutation relations. Poisson brackets between observables are promoted into commutators:
$$\{\cdot, \cdot\} \mapsto -i[\cdot, \cdot]. \tag{11}$$
This isn't a rigorous axiom, but rather a rule of thumb. As another example, the angular momentum $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ satisfies $\{L_i, L_j\} = \epsilon_{ijk}L_k$, which goes into the QM angular momentum commutation relations.

- Normal coordinates in SHO ($m = 1$): The simple harmonic oscillator is described by the Lagrangian $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$, which yields the EoM of $\ddot{x} - \omega^2 x = 0$. The conjugate momentum is $p = \partial L/\partial \dot{x} = \dot{x}$, and the Hamiltonian is $H = p^2/2 + \omega^2 x^2/2$. We'll perform a coordinate transformation to a system of **normal coordinates**, defined as:

$$a = \frac{\omega x + ip}{\sqrt{2\omega}} \qquad a^* = \frac{\omega x - ip}{\sqrt{2\omega}}. \tag{12}$$

  In normal coordinates, we can rewrite the Hamiltonian as:

$$H = \omega a a^*. \tag{13}$$

  The point of normal coordinates is that they are the analog of creation and annihilation operators. There's a way to write a classical Hamiltonian in terms of them, and if we want to quantize the theory we can **either** promote x and p to operators, **or** we can promote a and $a^\dagger$ to operators– these are equivalent. What should the commutation relations between a and $a^\dagger$ be? Well, let's evaluate the Poisson bracket:

$$\{a, a^*\} = \frac{\partial a}{\partial x}\frac{\partial a^*}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial a^*}{\partial x} = \sqrt{\frac{\omega}{2}}\frac{-i}{\sqrt{2\omega}} - \frac{i}{\sqrt{2\omega}}\sqrt{\frac{\omega}{2}} = -i \tag{14}$$

  So, to quantize the SHO, we take $[\hat{a}, \hat{a}^\dagger] = 1$. We see there are two equivalent quantization prescriptions:

  – Promote $x, p \mapsto \hat{x}, \hat{p}$ satisfying the canonical commutation relations $[x, p] = i$.

  – Promote $a, a^* \mapsto \hat{a}, \hat{a}^\dagger$ satisfying $[a, a^\dagger] = 1$.

- **Normal ordering**: The quantization prescription has one big ambiguity: what do we do with products of operators? If the Hamiltonian contains a term like xp, what do we do with it? c-numbers commute, so we can equally well write xp or px before quantization: either is fine, but will result in different terms in the quantum Hamiltonian, since $\hat{x}\hat{p} \neq \hat{p}\hat{x}$. We can see this in Eq. (13) with a and a^*.

  We need a convention to do the ordering, and we will pick one called **normal ordering**. We define a normal ordered product of operators a and $a^\dagger$ to put all the a's on the right, and all the $a^\dagger$'s on the left. We do this so that the ground state energy $\langle 0|H|0\rangle$ vanishes (in QFT, this takes care of the infinite ZPE that was presented in lecture). We denote this with colons surrounding the normal ordered expression:

$$H = \omega : a^\dagger a := \omega : a a^\dagger := \omega a^\dagger a. \tag{15}$$

- **Quantization prescription**:

  1. Identify the generalized coordinates of your system, q_i.
  2. From the Hamiltonian, derive the conjugate momentum p_i. These satisfy:

$$\{q_i, p_j\} = \delta_{ij} \tag{16}$$

  3. Quantize: promote q_i, p_i into **operators** $\hat{q}_i, \hat{p}_i$ that satisfy the canonical commutation relations:

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij} \tag{17}$$

Canonical quantization

- The quantization prescription for classical mechanics ports over to allow us to quantize a field theory. The whole problem in field theory, and the reason that this is a separate subject from quantum mechanics, is how you deal with quantizing systems that have an infinite number of degrees of freedom, but the general principle of promoting the coordinates and momenta to operators holds for field theory. In fact, a **quantum field** is just an operator-valued field.

- **Free scalar field**: As we've been studying in lecture, the free scalar field has Lagrangian density and EoM:
$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - m^2\phi^2 \qquad\qquad (\partial^2 - m^2)\phi(x) = 0. \tag{18}$$
Here the conjugate momentum to $\phi(x)$ is $\pi(x) = \dot{\phi}(x)$, which gives us the Hamiltonian density:
$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{19}$$
To quantize the field, we need to find the q's and the p's for this theory. The easiest way to do this is in momentum space, where the Fourier components $\tilde{\phi}(\boldsymbol{k}, t)$ obey the EoM for a harmonic oscillator:
$$(-\partial_t^2 + \omega_{\boldsymbol{k}}^2)\tilde{\phi}(\boldsymbol{k}, t) = 0. \tag{20}$$
The point of this is that we get a **harmonic oscillator** for every (continuous) momentum value– free theories in QFT are nothing but harmonic oscillators! So, the natural expansion for $\phi(\boldsymbol{x}, t)$ is in terms of the corresponding creation and annihilation operators for each mode:
$$\phi(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left(\hat{a}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_{\boldsymbol{k}}t} + \hat{a}_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\cdot x + i\omega_{\boldsymbol{k}}t}\right) \tag{21}$$
This is one of the central equations that we'll be using for canonical quantization of a scalar field theory.

- Conjugate momentum and canonical commutation relations: The usual relation holds to derive $\pi(\boldsymbol{x}, t)$:
$$\pi(\boldsymbol{x}, t) = \dot{\phi} = -i\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{k}}}{2}}\left(\hat{a}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_{\boldsymbol{k}}t} - \hat{a}_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\cdot x + i\omega_{\boldsymbol{k}}t}\right) \tag{22}$$
Like we saw in the previous section, the canonical quantization relations must hold when we quantize the theory. In this case, these take the form of:
$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}) \qquad\qquad [\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0 = [\pi(\vec{x}, t), \pi(\vec{y}, t)] \tag{23}$$

- What's novel in QFT? Creation and annihilation. In the SHO in QM, $a^\dagger|0\rangle$ just raises the energy state that we're using. In QFT, the interpretation of this is different. if I apply $a_{\boldsymbol{k}}^\dagger|0\rangle$ in QFT, we interpret it as creating a single-particle state with momentum $\boldsymbol{k}$.

- Derivation of the canonical commutation relations from $[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}}^\dagger] = (2\pi)^3\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')$ and $[a_{\boldsymbol{k}}, a_{\boldsymbol{k}}] = 0 = [a_{\boldsymbol{k}}^\dagger, a_{\boldsymbol{k}}^\dagger]$.

- Derivation of the momentum operator from
$$P^i = \int d^3\boldsymbol{x}\, T^{0i} = -\int d^3\boldsymbol{x}(\dot{\phi}\partial^i\phi^* + \dot{\phi}^*\partial^i\phi). \tag{24}$$