Welcome to QFT I!

- **Logistics**: all of the relevant course information can be found in the syllabus posted on Canvas. Here is my **contact info**, and the relevant times and locations for recitation and office hours:

TA	Email	Recitation	Location	Office Hours	OH Location
Patrick Oare	poare@mit.edu	Fridays, 2 PM	4-163	Tuesdays, 2 PM	8-308

  I'll try to be very responsive by email, so feel free to send me emails if there's anything you want to discuss or questions you have! Please also try to use the class **Piazza** page, as I'll be monitoring that to answer questions as they crop up.

- **Resources:** There are a lot of different textbooks out there which promise to give an "introduction to QFT"; some of them are decent, but most are indecipherable. I'd recommend trying a few of these out in your first few weeks and seeing which ones fit your learning style better– a lot of these books approach the same topics from different angles, and especially for a field as dense as QFT, it can be very valuable to have a few different ways to see the same problem. The main textbooks for this course are **Peskin & Schroeder** and **Weinberg**, and I also use **Schwartz** quite a bit. There's a blurb about the most of the specific books in the syllabus, and I'd encourage you to come and ask if you have any questions regarding the different resources!

Units in QFT

Fundamental constants provide a unit system in which we set them equal to 1. This course will use **natural units**, in which we set $\hbar = c = 1$. This is primarily done to simplify equations, since most relations in QFT have a large number of factors of $\hbar$ and c. This will look strange at first, and it is! What does $c = 1$ mean– how can you set a velocity equal to a dimensionless number? This definition sets two types of measurements equal to one another: time and length. $c = 3 \times 10^8$ m/sec, so $c = 1$ really means that 1 sec $= 3 \times 10^8$ meters. If I have a time measurement of 3 seconds, I can give it to you as 3 seconds, or equivalently as 9×10^9 meters; they're equivalent under the assumption $c = 1$!

As another example, the Compton wavelength of a mass m particle is defined as $\lambda_c = 1/m_e$ in natural units. I can measure λ_c as an inverse energy, since $E = mc^2 = m$, so typically you'll see λ_c quoted as an inverse energy like this, $\lambda_c = 1.9 \times 10^3\ \mathrm{GeV}^{-1}$. How do we get back to a value of λ_c in something we understand, like meters? We can **restore units** by multiplying this value with the appropriate factors of $\hbar$ and c, using $\hbar = 6.58 \times 10^{-16}\ \mathrm{eV} \cdot \mathrm{sec}$. Multiplying this with c gives us the conversion factor from GeV^{-1} to meters:

$$\hbar c = 0.2\ \mathrm{GeV} \cdot\ \mathrm{fm} = 1 \implies \lambda_c = (1.9 \times 10^3\ \mathrm{GeV}^{-1})(0.2\ \mathrm{GeV} \cdot \mathrm{fm}) = 380\ \mathrm{fm} = 3.8 \times 10^{-13}\ \mathrm{m} \quad (1)$$

Index notation

In this section, we'll briefly go over rotations in 2 dimensions as a vehicle for getting used to transformations and index notation.

- The space of proper (orientation preserving) rotational symmetries in 2 dimensions is $SO(2)$. One can represent the elements of $SO(2)$ as 2×2 matrices R_{ij} which are orthogonal, $R^T R = 1$, and have determinant 1, $\det R = 1$. An arbitrary $R \in SO(2)$ can be parameterized in the usual way:

$$R_{ij}(\theta, \hat{z}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2}$$

- Rotations are implemented on vectors in $\mathbb{R}^2$ by a matrix product with R_{ij}. We will use the **Einstein summation convention**: if an index is repeated, it is assumed to be summed over. A sum on indices is called a **contraction**. For example, the action of a rotation on a vector $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ is:

$$\boldsymbol{x} \mapsto R\boldsymbol{x} \quad \text{(vector notation)} \qquad\qquad x_i \mapsto \sum_j R_{ij} x_j \equiv R_{ij} x_j \quad \text{(index notation)}. \tag{3}$$

  Concretely, we can write this out as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R_{ij}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x_j} = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix} \tag{4}$$

  As another example, the previous condition of orthogonality, $R^T R = I$, translates into index notation as $R_{ki} R_{kj} = \delta_{ij}$, where δ_{ij} is the Kroenecker delta.

- Example: Orthogonal matrices R_{ij} do implement rotations because we can show they leave x^2 invariant:

$$x^2 = x_i x_i \mapsto (R_{ij} x_j)(R_{ik} x_k) = x_j \underbrace{(R^{\mathrm{T}})_{ji} R_{ik}}_{\text{matrix product}} x_k = x_j \delta_{jk} x_k = x^2. \tag{5}$$

- A **metric** is an inner product on a space, and gives us a notation of distance. In physics we represent metrics as symmetric matrices h_{ij}, under which the inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\langle \boldsymbol{x} | \boldsymbol{y} \rangle = h_{ij} x_i y_j$ (in matrix notation, this is $\boldsymbol{x}^{\mathrm{T}} h \boldsymbol{y}$). The Euclidean metric on $\mathbb{R}^2$ is the Kronecker delta $h_{ij} = \delta_{ij}$; a rotation can instead be defined as **a transformation which preserves the metric**, which is shown in the following equation:

$$R_{ki} \delta_{kl} R_{lj} = \delta_{ij} \qquad\qquad (R^{\mathrm{T}} I R = I \text{ in matrix notation}) \tag{6}$$

The Lorentz Group

- Special relativity tells us that in any reference frame with coordinates $x^\mu = (t, \boldsymbol{x})$, the spacetime interval $s^2 = -t^2 + \boldsymbol{x}^2$ is invariant. s is the norm of x^μ with respect to the **Minkowski metric**[1]:

$$g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \tag{7}$$

  The dot product between two vectors is $x \cdot y = g_{\mu\nu} x^\mu y^\nu$, so $s^2 = x \cdot x = g_{\mu\nu} x^\mu x^\nu$ is a norm squared. Greek letters $\mu \in \{0, 1, 2, 3\}$ denote spacetime indices and Latin letters $i \in \{1, 2, 3\}$ denote spatial indices.

- The **Lorentz group** is the set of symmetries of spacetime which preserve the metric $g_{\mu\nu}$. The total Lorentz group has 4 disconnected components: each component contains one of $\{1, P, T, PT\}$, where P is parity and T is time reversal. The component containing 1 is called the **proper orthochronous** subgroup and is denoted by $SO(1,3)$. An element $\Lambda \in SO(1,3)$ must satisfy (just like Eq. (6)):

$$\Lambda^\alpha{}_\mu g_{\alpha\beta} \Lambda^\beta{}_\nu = g_{\mu\nu} \tag{8}$$

  $SO(1,3)$ is a 6-dimensional Lie group, meaning any Lorentz transformation can be parameterized with 6 parameters: 3 rotation angles θ_i, and 3 boost parameters β_i.

- **Lorentz transformations**: rotations or boosts purely along one axis can be written out explicitly:

$$R(\hat{x}, \theta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \quad R(\hat{y}, \theta_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \quad R(\hat{z}, \theta_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

$$B(\hat{x}, \beta_1) = \begin{pmatrix} \cosh\beta_1 & \sinh\beta_1 & 0 & 0 \\ \sinh\beta_1 & \cosh\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B(\hat{y}, \beta_2) = \begin{pmatrix} \cosh\beta_2 & 0 & \sinh\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\beta_2 & 0 & \cosh\beta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B(\hat{z}, \beta_3) = \begin{pmatrix} \cosh\beta_3 & 0 & 0 & \sinh\beta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta_3 & 0 & 0 & \cosh\beta_3 \end{pmatrix} \tag{10}$$

  When multiple boost or rotation parameters are nonzero, one must use a matrix exponential to write Λ down as (we have also included an infinitesimal Lorentz transformation with $\beta_i, \theta_i \ll 1$):

$$\Lambda = \exp(i\beta_i K_i + i\theta_i J_i) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \approx 1 + \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu} + \mathcal{O}(\omega^2) \tag{11}$$

  Here K_i and J_i are antisymmetric 4×4 matrices which generate boosts and rotations[2]:

$$(J_i)_{jk} = -i\epsilon_{ijk} \qquad (K_i)_{0j} = \delta_{ij} = -(K_i)_{j0} \tag{12}$$

  and are packaged together covariantly as a *tensor* of 4×4 matrices $\mathcal{J}^{\mu\nu}$. $\omega_{\mu\nu}$ is an antisymmetric tensor which contains the parameters β_i and θ_i:

$$\mathcal{J}^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} \qquad \omega_{\mu\nu} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & 0 & \theta_3 & -\theta_2 \\ -\beta_1 & -\theta_3 & 0 & \theta_1 \\ -\beta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix} \tag{13}$$

[1]**WARNING**: Whenever you read a book, check the metric! In particle physics it's conventional to use the "mostly minus" convention, but in GR it's conventional to use the "mostly positive" convention with $\mathrm{diag}(-1, 1, 1, 1)$. Even the textbooks we use in this course will have different conventions: Weinberg uses mostly $+$, while Peskin uses mostly $-$.

[2]Explicitly written as matrices in Eqs. (10.14) and (10.15) of Schwartz.

- **Upper and lower indices**: Given a vector V^μ, one can form its dual vector V_μ by using the metric to lower its indices, $V_\mu = g_{\mu\nu}V^\nu$. Vectors with upper indices are called **contravariant**, and vectors with lower indices are **covariant**: under Lorentz transformations V^μ and V_μ transform in a dual way to one another. Some examples we will frequently use are:

$$x^\mu = (t, \boldsymbol{x}) \qquad p^\mu = (E, \boldsymbol{p}) \qquad \partial_\mu = (\partial_t, \boldsymbol{\nabla}). \tag{14}$$

- We can form multi-index **tensors** by combining upper and lower indices into one object $T^{\mu_1...\mu_k}{}_{\nu_1...\nu_\ell}$, which transforms under a Lorentz transformation Λ as:

$$T^{\mu_1...\mu_k}{}_{\nu_1...\nu_\ell} \mapsto \Lambda^{\mu_1}{}_{\alpha_1}...\Lambda^{\mu_k}{}_{\alpha_k}\Lambda_{\nu_1}{}^{\beta_1}...\Lambda_{\nu_\ell}{}^{\beta_\ell}T^{\alpha_1...\alpha_k}{}_{\beta_1...\beta_\ell}. \tag{15}$$

  The number of indices $k+\ell$ is called the **rank** of the tensor. Vectors V^μ and V_μ are rank 1 tensors. Examples of rank 2 tensors are the metric $g_{\mu\nu}$, the stress-energy tensor $T_{\mu\nu}$, and the field strength $F_{\mu\nu}$.

- A quantity is **Lorentz invariant** if it is the same in all reference frames. A general rule is that to form a Lorentz invariant, every upper index you see must be contracted with a lower index, and every lower index with an upper index. Quantities like $x\cdot\partial = x^\mu\partial_\mu$, ∂^2, and $p^\mu\partial^\nu F_{\mu\nu}$ are Lorentz invariant. In particular, any **dot product or square of vectors is invariant**. An consequence of this is that *for a massive particle, p^2 will always equal m^2*, since in its rest frame $p^\mu = (m, \boldsymbol{0})$.

- A quantity that is **Lorentz covariant** will change in different reference frames, but in a way that respects the metric (all indices must be contracted in a Lorentz invariant way), for example $\partial_\mu T^{\mu\nu}$.

Classical Field Theory

- A classical **field** is a function on spacetime. ϕ can take different types of values, which define different types of fields. Some examples you've seen are scalar fields $\phi(x)$, three-vector fields like $\boldsymbol{E}(x)$, four-vector fields like $A_\mu(x)$, and tensor fields like $F_{\mu\nu}(x)$. The **Lagrangian** L for a field theory can be written as an integral of a **Lagrangian density** $\mathcal{L}$, and from it we can construct the **action** S:

$$L[\phi] = \int d^3\boldsymbol{x}\; \mathcal{L}(\phi, \partial_\mu\phi) \qquad S[\phi] = \int dt\; L[\phi] = \int d^4x\; \mathcal{L}(\phi, \partial_\mu\phi) \tag{16}$$

- **Field transformations**: A field $\phi(x)$ transforms under a Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)$ to $\phi'(x)$ such that at the transformed point $x' = \Lambda x$, the field remains the same, $\phi'(x') = \phi(x)$. This transformation law is written out as:

$$\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1}x). \tag{17}$$

  The field can have more indices, for example the electromagnetic potential $A_\mu(x)$ or field strength $F_{\mu\nu}(x)$ from E&M. In this case, each of the external indices transforms in the appropriate way. For an arbitrary tensor field:

$$T^{\mu_1...\mu_k}{}_{\nu_1...\nu_\ell}(x) \mapsto \Lambda^{\mu_1}{}_{\alpha_1}...\Lambda^{\mu_k}{}_{\alpha_k}\Lambda_{\nu_1}{}^{\beta_1}...\Lambda_{\nu_\ell}{}^{\beta_\ell}T^{\alpha_1...\alpha_k}{}_{\beta_1...\beta_\ell}(\Lambda^{-1}x). \tag{18}$$

  The way that a field transforms under Lorentz transformations classifies if we call it a scalar, vector, or tensor. For example, we can have a two-component scalar field $\phi_a(x)$, since a is not a spacetime index; under a Lorentz transformation, $\phi_a(x)$ just transforms into itself.

- From the principle of least action, once can derive the **Euler-Lagrange equations** for a field:
$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}. \tag{19}$$
These equations govern the evolution of the field ϕ.

- A **symmetry** of a theory is a transformation which leaves the action invariant. Note that this *does not imply* that the Lagrangian density is invariant, but rather that it can change by a total derivative. If this is the case, **Noether's theorem** tells us that there is a conserved current and charge associated with the symmetry. Suppose the transformation $\phi \mapsto \phi + \alpha \Delta \phi$ is a symmetry of the theory, and that the Lagrangian density changes by a total derivative, $\mathcal{L} \mapsto \mathcal{L} + \alpha \partial_\mu \mathcal{F}^\mu$. The **conserved current** is:
$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - \mathcal{F}^\mu \qquad \qquad \partial_\mu j^\mu = 0. \tag{20}$$
Assuming there is no net charge flow at spatial infinity, this yields a **conserved charge** Q given by:
$$Q = \int d^3\boldsymbol{x}\; j^0(x) \qquad \qquad \frac{dQ}{dt} = -\int d^3\boldsymbol{x}\; \nabla \cdot \boldsymbol{j} = 0. \tag{21}$$

- **Example**: two decoupled scalar fields. Consider the following field theory, where $\phi(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \end{pmatrix}$:
$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_a)(\partial^\mu \phi_a) - \frac{1}{2} m^2 \phi_a \phi_a. \tag{22}$$
Here $a = 1, 2$ is just an extra index labeling which copy of the scalar field we're looking at. To find the EoM for this field, we can evaluate the usual derivatives, being careful to tack on an "a" index:
$$\frac{\partial \mathcal{L}}{\partial \phi_a} = -m^2 \phi_a \tag{23}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = \frac{\partial}{\partial(\partial_\mu \phi_a)} \left(-\frac{1}{2} g^{\alpha\beta} (\partial_\alpha \phi_b)(\partial_\beta \phi_b) \right) = -\frac{1}{2} g^{\alpha\beta} \underbrace{\left(\delta^\mu_\alpha \delta_{ab} (\partial_\beta \phi_b) + (\partial_\alpha \phi_b) \delta^\mu_\beta \delta_{ab} \right)}_{\text{product rule}} = -\partial^\mu \phi_a \tag{24}$$
Differentiating Eq. (24) and setting it equal to Eq. (23) gives us the EoM $(\partial^2 - m^2)\phi_a(x) = 0$, which is just two copies of the Klein-Gordon equation.

This has an $SO(2)$ symmetry, since only the norm squared of ϕ_a and $\partial_\mu \phi_a$ enter the Lagrangian. Let's derive the Noether current for this symmetry by considering a rotation by angle $\alpha \ll 1$:
$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} + \mathcal{O}(\alpha^2) \tag{25}$$
$$\implies \begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \end{pmatrix} = \begin{pmatrix} -\phi_2 \\ \phi_1 \end{pmatrix} \tag{26}$$
Since the Lagrangian is invariant, $\mathcal{F}^\mu = 0$, so we can construct the Noether current:
$$j^\mu = (\partial^\mu \phi_1)\phi_2 - \phi_1(\partial^\mu \phi_2) \qquad \qquad Q = \int d^3\boldsymbol{x}\; (\dot{\phi}_1 \phi_2 - \phi_1 \dot{\phi}_2) \tag{27}$$
This is an angular momentum! In the 1D field theory case, $\phi_1(x), \phi_2(x) \leftrightarrow x(t), y(t)$, and $Q = \dot{x}y - x\dot{y} = \boldsymbol{x} \times \dot{\boldsymbol{x}}$, which is the angular momentum of a mass $m = 1$ particle at position $(x(t), y(t))$.