The Lorentz and Poincaré Groups

Patrick Oare

1 The Lorentz Group and Algebra

Let $g_{\mu\nu} := diag(1, -1, -1, -1)$ be the standard Minkowski metric. The **Lorentz group**, denoted $SO(1,3)$, is the subgroup of $M_{4\times 4}(\mathbb{R})$ which leaves g invariant under conjugation, i.e. a 4 by 4 matrix Λ is in $SO(1,3)$ iff:

$$\Lambda^T g \Lambda = g \tag{1}$$

As you know from Special Relativity, flat spacetime has a Minkowski metric and elements of the Lorentz group boost states into different frames. As a topological space, the Lorentz group is disconnected: it has four connected components, which correspond to where the symmetries of **parity**, $P = diag(1, -1, -1, -1)$, and **time reversal**, $T = diag(-1, 1, 1, 1)$ lie. We call a Lorentz transformation $\Lambda \in SO(1,3)$ **proper** if $det(\Lambda) = 1$ and **orthochronous** if $\Lambda^0_0 > 1$, and we can split $SO(1,3)$ into its connected components based on where these symmetries lie, as depicted in Figure 1.

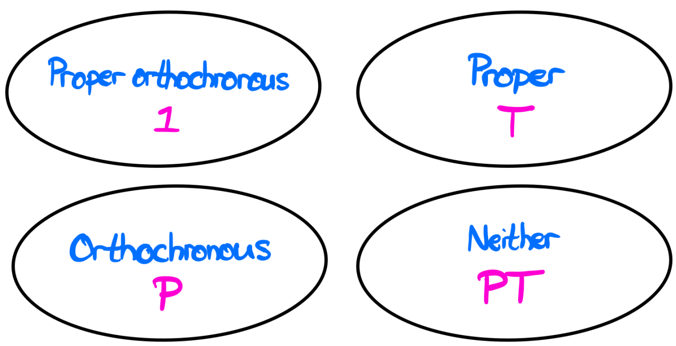

Figure 1: The four connected components of the Lorentz group. The proper orthochronous component is in the neighborhood of 1, and thus the only elements that can be generated as exponentials of the Lie algebra lie in this component. Each component contains exactly one of $P^a T^b$, where $a, b \in \{0, 1\}$.

When we study the Lie algebra $\mathfrak{so}(1,3)$, we are only generating the proper orthochronous subset of the Lorentz group $SO(1,3)^+$. However, it is still very helpful to understand the structure of this subset of $SO(1,3)$, as any arbitrary Lorentz transformation is a product (properorthochronous) $\times$ $\mathrm{P}^{\mathrm{a}} \times \mathrm{T}^{\mathrm{b}}$ for $a, b \in \{0, 1\}$. The algebra $\mathfrak{so}(1,3)$ **has 6 generators**, which are expressed in two different ways. The tensorial way to arrange these generators is to write them as an antisymmetric rank 2 tensor $\mathcal{J}_{\mu\nu}$. This tensor can be expressed as the following operator acting on wavefunctions:

$$\mathcal{J}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2}$$

and we can thus write any proper orthochronous Lorentz transformation $\Lambda \in SO(1,3)^+$ as:

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \tag{3}$$

The factor of $\frac{1}{2}$ is conventional because $\mathcal{J}^{\mu\nu}$ is antisymmetric. Note also that $\omega_{\mu\nu}$ can also be taken to be antisymmetric, as the symmetric part will vanish when contracted with $\mathcal{J}^{\mu\nu}$. Written with this generator, the Lorentz algebra is:

$$[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = i(g_{\mu\sigma}\mathcal{J}_{\nu\rho} + g_{\nu\rho}\mathcal{J}_{\mu\sigma} - g_{\mu\rho}\mathcal{J}_{\nu\sigma} - g_{\nu\sigma}\mathcal{J}_{\mu\rho}) \tag{4}$$

The other way the generators of $SO(1,3)$ are conventionally written is as a **angular momentum generator** J_i and a **boost generator** K_i for $i \in \{1,2,3\}$. This decomposition comes from separating the timelike parts of $\mathcal{J}_{\mu\nu}$ from the spacelike parts of $\mathcal{J}_{\mu\nu}$. The generators are defined as:

$$J^i := \frac{1}{2}\epsilon^{ijk}\mathcal{J}_{jk} \qquad\qquad K^i := \mathcal{J}^{0i} \tag{5}$$

which clearly allows us to expand the tensor as (we also expand $\omega_{\mu\nu}$ which will simplify things later):

$$\mathcal{J}_{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix} \qquad \omega_{\mu\nu} = \begin{pmatrix} 0 & \lambda^1 & \lambda^2 & \lambda^3 \\ -\lambda^1 & 0 & \theta^3 & -\theta^2 \\ -\lambda^2 & -\theta^3 & 0 & \theta^1 \\ -\lambda^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix} \tag{6}$$

where the three boost parameters are $\vec{\lambda}$ and the rotation parameters are $\vec{\theta}$. They satisfy the algebra:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k \\ [J_i, K_j] &= i\epsilon_{ijk}K_k \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k \end{aligned} \tag{7}$$

which makes it clear that the elements J_i generate **rotations** (note their algebra is $\mathfrak{so}(3)$) and hence are generators of $SO(3)$. Finally, with these generators we can write an arbitrary Lorentz transformation as parameterized by $\vec{\lambda}$ and $\vec{\theta}$:

$$\Lambda = \exp\left(-i\vec{\theta}\cdot\vec{J} - i\vec{\lambda}\cdot\vec{K}\right) \tag{8}$$

The boost generators are more complicated, and do not generate anything as simple as $SO(3)$. However, the algebra can be simplified dramatically by taking a specific linear combination of the generators:

$$J_i^{\pm} := \frac{1}{2}(J_i \pm iK_i) \tag{9}$$

These $\{J_i^{\pm}\}$ also generate $\mathfrak{so}(1,3)$. More importantly, their algebra is easier to deal than how the Lorentz algebra was previously cast:

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm} \qquad\qquad [J_i^{\pm}, J_j{\mp}] = 0 \tag{10}$$

This makes it explicit that $\{J_i^+\}$ and $\{J_i^-\}$ each generate their own independent $\mathfrak{su}(2)$ subalgebra of $\mathfrak{so}(1,3)$. Because of this, the entire algebra $\mathfrak{so}(1,3)$ has a decomposition as a sum:

$$\mathfrak{so}(1,3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{11}$$

In the next section we will use this decomposition of the Lorentz algebra to classify all the irreps of the Lorentz group.

2 Representation theory of the Lorentz group

Because the Lorentz group implements Lorentz transformations, understanding its representation theory is crucial. Fortunately, you've likely been using representations of the Lorentz group without even knowing it. In particular, scalars, vectors, and tensors transform in different representations of the Lorentz group. Consider a vector V_μ. This lives in $\mathbb{R}^4$, and we allow a Lorentz transformation $\Lambda \in SO(1,3)$ to act on the vector as:

$$V^\mu \mapsto \Lambda^\mu_\nu V^\nu \tag{12}$$

Suppressing the indices, this looks like $V \mapsto D(\Lambda)V$, where $D(\Lambda)$ has the components $\Lambda^{\mu\nu}$, i.e. D is simply the identity. Thus whenever we work with vectors in special relativity, we are simply using the **fundamental representation** of the Lorentz group. Written out explicitly, this representation is $id : SO(1,3) \to Aut(\mathbb{R}^4)$, $\Lambda \mapsto \Lambda_{\mu\nu}$. Here we are viewing $\Lambda \in SO(1,3)$ as an abstract element of a group (which is defined as a matrix group), and we explicitly view $id(\Lambda)$ as a 4×4 matrix which has components $\Lambda^{\mu\nu}$. We will denote this fundamental representation by $\mathbf{4}$, its dimension.

In a similar way, scalars and tensors are also representations of the Lorentz group. Scalars obviously live in the singlet representation $\mathbf{1}$ since a scalar ϕ does not transform, i.e $D(\Lambda)\phi = \phi$. 2-tensors $T^{\mu\nu}$ live in the tensor product representation $\mathbf{16} = \mathbf{4} \otimes \mathbf{4}$, since under a Lorentz transformation $\Lambda \in SO(1,3)$ they transform as:

$$T^{\mu\nu} \mapsto \Lambda^\mu_\rho \Lambda^\nu_\sigma T^{\rho\sigma} \tag{13}$$

This can be written out as a matrix equation $T \mapsto D(\Lambda)T$, where T is viewed as a 16 dimensional vector and $D(\Lambda) = \Lambda \otimes \Lambda \in Aut(\mathbb{R}^{16})$ (viewing Λ as a matrix) is a 16×16 dimensional matrix. Thus 2-tensors $T^{\mu\nu}$ live in the representation $\mathbf{16}$.

Unlike the representations $\mathbf{1}$ and $\mathbf{4}$ that we have seen previously, the representation $\mathbf{16}$ is a reducible representation. Let (D_{16}, V_{16}) be this representation. We can define 3 subspaces of V_{16} as follows:

$$\begin{aligned} W &:= \left\{ \frac{1}{4} T^\alpha_\alpha g^{\mu\nu} : T^{\mu\nu} \in V_{16} \right\} \\ A &:= \left\{ \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) : T^{\mu\nu} \in V_{16} \right\} \\ S &:= \left\{ \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) - \frac{1}{4} T^\alpha_\alpha g^{\mu\nu} : T^{\mu\nu} \in V_{16} \right\} \end{aligned} \tag{14}$$

These are respectively the subspaces of traces, antisymmetric tensors, and traceless symmetric tensors. Note that $dim(W) = 1$, $dim(A) = 6$, and $dim(S) = 9$, so we will denote them respectively by $\mathbf{1}$, $\mathbf{6}$, and $\mathbf{9}$.

The space V_{16} of all 2-tensors splits as a direct sum of these subspaces, as each tensor $T^{\mu\nu}$ can be written as a sum of a symmetric and antisymmetric component, and the symmetric component can further be split into a trace part and a traceless part. Thus we have the decomposition:

$$V_{16} = W \oplus A \oplus S \tag{15}$$

which is written as $\mathbf{16} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}$ if we denote them by their dimensionalities. Furthermore, each of these subspaces is invariant under the action of the Lorentz group because tensor transformations preserve symmetry and antisymmetry, and trace is a Lorentz singlet. The representation $\mathbf{6}$ of antisymmetric tensors is the adjoint representation because the Lorentz group has 6 generators, and we will later see its decomposition into irreps. The representation $\mathbf{9}$ of symmetric traceless tensors is irreducible. **9** plays an important role in QFT, as we will often attempt to decompose tensor operators into a sum of tensors which live in irreps of the Lorentz group[1].

We now turn to the question of the irreps of the Lorentz group. From the decomposition in Eq. (11), we can classify **all** the irreps of $\mathfrak{so}(1,3)$ using the decomposition theorem for irreps of a direct sum, which tells us that the irreps of a sum of Lie algebras are exactly the tensor products of their individual irreps. Although we are summing two algebras together, their corresponding irreps are tensor products, not sums. We previously classified the irreps of $\mathfrak{su}(2)$ completely and showed we could label each irrep with a (half) integer $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$, and the corresponding irrep π_j has dimension $2j+1$. Thus, **we can label all the irreps of $SO(1,3)$ by a pair**:

$$(j_+, j_-) \tag{16}$$

with $j_+, j_- \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$, and (j_+, j_-) denoting the tensor product representation $\pi_{j_+} \otimes \pi_{j_-}$. We also see that the irrep (j_+, j_-) has a dimensionality $D^{p,q}$ given by:

$$D^{j_+,j_-} = dim(\pi_{j_+} \otimes \pi_{j_-}) = (2j_+ + 1)(2j_- + 1) \tag{17}$$

and furthermore, that the irrep of dimension **k** is the unique such irrep of that dimension.

The fundamental irrep **4** can be denoted in this convention as $(\frac{1}{2}, \frac{1}{2})$, and the irrep **9** of symmetric traceless tensors is denoted $(1,1)$. If $j_+ + j_-$ is an integer we call the irrep (j_+, j_-) a **tensor representation** and if $j_+ + j_-$ is a half integer we call the irrep a **spinor representation**. The reason for this comes from QFT. If a field lives in a tensor irrep, then it will have a Lorentz index, which is why a spin 1 particle like the photon A_μ will have a single Lorentz index.

On the other hand, a spinor representation (j_+, j_-) will have spinor indices and not Lorentz indices. The example of this which should come to mind is Dirac spinors. The representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are equivalent the spin $\frac{1}{2}$ representations of $\mathfrak{su}(2)$, and so they are 2 dimensional and J_i^+ and J_i^- are represented on each by either the Pauli matrices, or zero. Physically, these correspond to right and left handed spinors ψ_L and ψ_R, which is why these spinors have 2 components. The **Dirac representation** of $SO(1,3)$ (also called the **bispinor** representation) is the sum $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, and this is the representation that we typically use to study spin $\frac{1}{2}$ particles in QFT. The basis that shows this decomposition of $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ explicitly is the Weyl basis, which is why Weyl spinors decouple into 2 dimensional left and right spinors.

Of particular interest is the adjoint representation **6**. Although $D^{\frac{1}{2},1} = 6$, it is important to note that $(\frac{1}{2}, 1)$ and **6** are different representations. **6** can actually be decomposed into a direct sum $\mathbf{6} = (\mathbf{1}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{1})$, and so in fact 6 is a reducible tensor representation. This should make physical sense, because the antisymmetric field strength tensor $F_{\mu\nu}$ must live in a tensor representation because it is constructed from the photon field A_μ. One can decompose $F_{\mu\nu}$ into this invariant decomposition $(1,0) \oplus (0,1)$ by taking specific linear combinations of $\vec{E}$ and $\vec{B}$ which rotate into themselves under boosts, as these spatial vectors live in the representations $(1,0)$ and $(0,1)$ which each have dimension 3.

Now, consider a massive (or massless) vector field $A_\mu(x)$ like the photon. Because it is a vector field, it lives in the fundamental **4** of the Lorentz group. From QFT, we know that $A_\mu(x)$ will describe a spin 1 particle, which as 3 degrees of freedom if massive and 2 degrees of freedom if massless. How do these degrees of freedom relate to the dimension of the representation **4**? We will see this in the next section when we study the Poincaré group: spin embeds itself into fields

[1] In quantum mechanics, this procedure carried out for Euclidean tensors V_{ij} gives a similar decomposition, and the corresponding decomposition of symmetric and traceless tensors gives an irreducible tensor operator for which you can apply the Wigner-Eckart theorem (although because the dimensions are different this subspace is only 5 dimensional in QM)

via the **little group**, which is a subset of symmetries of the Poincaré group. In particular, the fundamental representation $\mathbf{4}$ of the Lorentz group decomposes into a sum $\mathbf{1}\oplus\mathbf{3}$ of $SO(3)$, the little group for massive spin 1 particles, and as these irreps correspond to angular momentum 0 and 1, we are able to embed scalar and spin 1 particles into this particular representation of the Lorentz group.

To sum up our discussion, we will enumerate some of the different irreps and their names in a table.

Name	(j_+, j_-) label	Dimension	Irrep?
Singlet	$(0,0)$	1	Y
Left Weyl (ψ_a)	$(\frac{1}{2},0)$	2	Y
Right Weyl ($\psi^{\dot{a}}$)	$(0,\frac{1}{2})$	2	Y
Dirac (bispinor)	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	4	N
Vector (V_μ)	$(\frac{1}{2},\frac{1}{2})$	4	Y
Adjoint (curvature $F_{\mu\nu}$)	$(1,0)\oplus(0,1)$	6	N
—	$(1,\frac{1}{2})$	6	Y
Symmetric Tensor ($S_{\mu\nu}$)	$(1,1)$	9	Y

Table 1: Low dimensional representations of the Lorentz group. In the next section, we will further study the spin $\frac{1}{2}$ representations of the Lorentz group, which give us a good example of spinor representations.

2.1 Weyl Spinors

Spin $\frac{1}{2}$ representations of the Lorentz group are important because they provide an example of spinor representations, which fermions live in. In particular, **the fundamental fermion fields in the Standard Model are all left-handed Weyl or right-handed Weyl spinors**, and so to understand the Standard Model it is important to understand how Weyl spinors work.

Before diving into the indices that we will use to study these representations, a good starting place is to see what Lorentz transformations actually look like in the $D_L := (\frac{1}{2},0)$ and $D_R := (0,\frac{1}{2})$ representations of $SO(1,3)$. This is done by seeing where the generators are sent. For the left handed representation, we have $j_+ = \frac{1}{2}$ and $j_- = 0$, so we get two equations:

$$D_L(J_i^+) = \frac{1}{2}\left[D_L(J_i) + iD_L(K_i)\right] = \frac{1}{2}\sigma_i \qquad D_L(J_i^-) = \frac{1}{2}\left[D_L(J_i) - iD_L(K_i)\right] = 0 \tag{18}$$

These equations imply that **in the left-handed Weyl irrep**, the generators are mapped to:

$$(J_i)_L = \frac{1}{2}\sigma_i \tag{19}$$

$$(K_i)_L = -i\frac{1}{2}\sigma_i \tag{20}$$

The notation here uses the subscript L to denote that J^i or K^i is in the left-handed Weyl representation, i.e. $J_L^i = D_L(J^i)$, and applies likewise for K^i and right-handed spinors. This can be extended to show how a left-handed spinor ψ_L transforms under the Lorentz group. In the right handed representation D_R, $D_R(J_i^+) = 0$ and $D_R(J_i^-) = \frac{1}{2}\sigma_i$, so the boost generator flips sign:

$$(J_i)_R = \frac{1}{2}\sigma_i \tag{21}$$

$$(K_i)_R = i\frac{1}{2}\sigma_i \tag{22}$$

To perform a Lorentz transformation on a Weyl spinor in either the D_L or D_R representations, one just substitute how the generators J_i and K_i look in the corresponding representation, and then apply Eq. (8). We often consider (see Srednicki) these generators packaged together in their tensor form as well:

$$S_L^{\mu\nu} = D_L(\mathcal{J}^{\mu\nu}) \qquad\qquad S_R^{\mu\nu} = D_R(\mathcal{J}^{\mu\nu}) \tag{23}$$

and so for an arbitrary Weyl spinor ψ_L or ψ_R, S_L and S_R generate the corresponding Lorentz transformation Λ with parameters $\omega_{\mu\nu}$ as:

$$\psi_L^a(x) \mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}S_L^{\mu\nu}\right)^a_{\ b}\psi^b(\Lambda^{-1}x) \qquad \psi_R^{\dot{a}}(x) \mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}S_R^{\mu\nu}\right)^{\dot{a}}_{\ \dot{b}}\psi^{\dot{b}}(\Lambda^{-1}x) \tag{24}$$

Compactly, the generators S_L and S_R in the left/right-handed Weyl irreps are related by negation and conjugation:

$$(S_R^{\mu\nu})_{\dot{a}}^{\ \dot{b}} = -[(S_L^{\mu\nu})_a^{\ b}]^* \tag{25}$$

Now we can discuss the indices (we have already cheated a bit by dotting the S_R generator's indices). Conventionally, we use undotted indices a to denote the $D_L = (\frac{1}{2}, 0)$ representation, and dotted indices $\dot{a}$ to denote the $D_R = (0, \frac{1}{2})$ representation. These indices are helpful because **a Lorentz invariant can only be formed by contracting the same type of indices**, i.e. if ϕ^a and $\chi_{\dot{b}}$ are a left and right handed spinor respectively then $\phi^a\chi_{\dot{a}}$ is *not Lorentz invariant*.

Hermitian conjugation interchanges dotted and undotted indices, that is, it maps left handed spinors to right handed spinors and vice versa. This is because of how the generators are defined as $J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)$. Since the J_i and K_i are hermitian, $(J_i^{\pm})^\dagger = J_i^{\mp}$, which implies that $\dagger$ maps $(\frac{1}{2}, 0)$ into $(0, \frac{1}{2})$, and vice versa. If ψ_L is a left handed Weyl spinor, then it has an undotted index ψ_L^a. However, its Hermitian conjugate is a right handed spinor, and so has a dotted index, $(\psi_L^\dagger)^{\dot{a}}$.

The Clebsch-Gordan theory for $SU(2)$ carries over reasonably nicely to spin 1/2 particles. As an example, consider a tensor $C_{\alpha\beta}$ which has two undotted indices, and therefore lives in $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$. We wish to see if this is irreducible, or if it can be decomposed into a sum of terms which each do not mix with one another. Since the first component has the relation $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ in $SU(2)$, this carries over to our current situation. Note that the singlet 0 is antisymmetric and the triplet 1 is symmetric, so that imples that we can decompose any tensor C_{ab} as:

$$C_{ab} = \epsilon_{ab}D + G_{ab} \tag{26}$$

where ϵ_{ab} is the totally antisymmetric 2d Levi-Civita symbol:

$$\epsilon^{12} = -\epsilon^{21} = 1 = \epsilon_{21} = -\epsilon_{12} \tag{27}$$

and G_{ab} is a symmetric tensor, i.e. in matrix form:

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad\qquad \epsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{28}$$

Note that $\epsilon^{ab} = -\epsilon_{ab}$, so it is important to keep the upper and lower indices in mind when working with this symbol. **The Levi-Civita symbol ϵ is also used to raise and lower indices on spinors**, i.e. $\psi_L^a = \epsilon^{ab}(\psi_L)_b$.

2.2 Dirac and Majorana spinors

Although left/right handed Weyl spinors are the simplest types of spin $\frac{1}{2}$ objects we can consider, we often work with different types of fermions and larger dimensional representations. Another spin $\frac{1}{2}$ representation that is frequently used is the **Dirac (bispinor) representation** of $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. Spinors are often introduced in this representation as it is a bit easier to get here from the physics of quantum mechanics than to start with the representation theory of the Lorentz group. Dirac fermions are most easily expanded in the **Weyl basis**, in which the Dirac spinor is a four component spinor made by stacking a left-handed and right-handed spinor on top of one another. Let us have two left-handed fields, χ_a and ξ_a (note they both start with a lowered index). Then the **Dirac spinor** containing these fields is:

$$\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix} \tag{29}$$

Care must be made when working with these equations in spinor form. Because indices are raised with $\epsilon^{ab} = i\sigma^2$, when explicitly written out in components, this implies the Dirac spinor is formed from χ_a and ξ_a as:

$$\Psi = \begin{pmatrix} \chi_a \\ \epsilon^{\dot{a}\dot{b}}\xi^\dagger_{\dot{b}} \end{pmatrix} = \begin{pmatrix} \chi \\ i\sigma^2\xi^* \end{pmatrix} \tag{30}$$

since $\epsilon^{ab} = i\sigma^2$ and $\epsilon_{ab} = -i\sigma^2$. The four γ matrices γ^μ can be encoded as follows:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{31}$$

When working with a Dirac fermion Ψ, we typically will not use the Hermitian conjugate $\Psi^\dagger$, but rather consider the **Dirac conjugate** of Ψ:

$$\overline{\Psi} = \Psi^\dagger\beta \tag{32}$$

where as a matrix, $\beta = \gamma^0$ (we use β here because technically γ^0 is part of a four vector γ^μ). To see why we would consider this, note that in the Weyl basis the difference between $\Psi^\dagger$ and $\overline{\Psi}$ is a swapping of chiral components:

$$\overline{\Psi} = \begin{pmatrix} \xi^a & \chi^\dagger_{\dot{a}} \end{pmatrix} \qquad \Psi^\dagger = \begin{pmatrix} \chi^\dagger_{\dot{a}} & \xi^a \end{pmatrix} \tag{33}$$

This is important because $\chi^\dagger$ and χ cannot be contracted and we cannot form a Lorentz invariant from them:

$$\Psi^\dagger\Psi = \chi^\dagger\chi + \xi^\dagger\xi = (\chi^\dagger)_{\dot{a}}\chi_a + \xi^a\xi^{\dagger\dot{a}} \tag{34}$$

as can be clearly seen because χ and $\chi^\dagger$ have different indices, one dotted and one undotted. If we instead consider $\overline{\Psi}$, we see that the correct left/right handed Weyl spinors are contracted with one another, i.e. that the dotted and undotted indices agree:

$$\overline{\Psi}\Psi = \xi^a\chi_a + \chi^\dagger_{\dot{a}}\xi^{\dagger\dot{a}} \tag{35}$$

Note it is simpler to keep track of indices, as in the SM we will often just take a right handed Weyl spinor (for example e_R) and then we will be using ξ instead of $\xi^\dagger$. When added to the Lagrangian, this is called the **Dirac mass** term. The full Lagrangian for a Dirac field is:

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{36}$$

which when written in the Weyl basis, splits into the two Weyl Lagrangians, one for the left-handed field, and another for the right-handed field.

In practice because most of the computational tools we know are for Dirac spinors, we often with to project Dirac spinors onto a state of definite chirality. This allows us to embed two-component Weyl spinors into four-component Dirac spinors. Projectors onto the left/right handed subspaces are given by:

$$P_L = \frac{1}{2}(1-\gamma_5) \qquad\qquad P_R = \frac{1}{2}(1+\gamma_5) \tag{37}$$

For Standard Model calculations, these projectors will always be inserted in front of the fermion fields which are being used, since the SM only contains Weyl fermions. Using $\{\gamma_\mu, \gamma_5\} = 0$, one can rearrange the projectors as this implies $\gamma^\mu P_L = P_R \gamma^\mu$, and this self-consistently connects spinors with the correct handedness as dictated by the indices.

The generators $\mathcal{J}^{\mu\nu}$ can be considered in the Dirac representation. In this representation, they can be expanded using the $\sigma_{\mu\nu}$ matrices:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{38}$$

The generator $J^{\mu\nu}$ in the bispinor representation can be simply expressed as:

$$D_{\text{bispinor}}(\mathcal{J}^{\mu\nu}) = \frac{1}{2}\sigma^{\mu\nu} \tag{39}$$

which explains why $\sigma^{\mu\nu}$ plays such an important role in Dirac algebra computations. This generator of the bispinor representation can also be related to the two generators of the LH and RH representations:

$$\frac{1}{2}\sigma^{\mu\nu} = \begin{pmatrix} (S_L^{\mu\nu})_a^{\ c} & 0 \\ 0 & -(S_R^{\mu\nu})^{\dot{a}}_{\ \dot{c}} \end{pmatrix} \tag{40}$$

As for mass terms, Weyl spinors can form mass terms too. However, this is a bit surrounded in technical jargon. When we discuss massive chiral fermions, we often use **Majorana spinors**, which allows us to embed chiral Weyl fermions into the bispinor representation. Given a left-handed Weyl fermion ψ in the $(\frac{1}{2}, 0)$ representation, we have a corresponding right-handed spinor $\psi^\dagger_{\dot{a}}$. The associated Majorana spinor Ψ is then constructed as a Dirac spinor using ψ and $\psi^\dagger$:

$$\Psi = \begin{pmatrix} \psi_a \\ \psi^{\dagger\dot{a}} \end{pmatrix} = \begin{pmatrix} \psi \\ i\sigma^2\psi^* \end{pmatrix} \tag{41}$$

Here this is exactly how we defined a Dirac spinor, but the right handed component is $\psi^{\dagger\dot{a}} = \epsilon^{\dot{a}\dot{b}}\psi^\dagger_{\dot{b}} = \epsilon\psi^*$ when we suppress the indices. A Majorana fermion has **two degrees of freedom and four components**, because it is in the bispinor representation but corresponds exactly to a chiral Weyl fermion with 2 components. From a Majorana spinor, we can form a mass term (we can really form this for any Weyl spinor, but conventionally we call it a **Majorana mass**):

$$m\left(\psi\psi + \psi^\dagger\psi^\dagger\right) = m\left(\epsilon^{ab}\psi_a\psi_b + \epsilon^{\dot{a}\dot{b}}\psi^\dagger_{\dot{a}}\psi^\dagger_{\dot{b}}\right) = m\left(\psi^T i\sigma^2\psi + \psi^{*T} i\sigma^2\psi^*\right) \tag{42}$$

To tell if a Dirac spinor is Majorana, one can see if it satisfies the **reality constraint**: a Dirac spinor Ψ is Majorana iff it equals its charge conjugate:

$$\Psi = \Psi^{\mathrm{C}} \tag{43}$$

where the charge conjugate essentially switches the corresponding left and right handed fields inside the Dirac spinor. Charge conjugation is defined rigorously in the next section, but for now it suffices to note that for a Dirac spinor made up of left-handed Weyl spinors χ and ξ:

$$\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger \dot{a}} \end{pmatrix}, \tag{44}$$

its charge conjugate Ψ^{C} is:

$$\Psi^{\mathrm{C}} = \begin{pmatrix} \xi_a \\ \chi^{\dagger \dot{a}} \end{pmatrix}. \tag{45}$$

This shows us that a Dirac spinor is Majorana iff $\chi = \xi$, i.e. if it is composed of the same spinor. If Ψ is not Majorana, then its upper and lower components correspond to different particles, and not simply conjugates of the same particle.

2.3 Spinor indices and invariant symbols

Since we now have the basics of spinor indices set up, we will discuss how to work with them. We will begin by discussing the ϵ symbol in more detail. The Levi-Civita tensor is also called an **invariant symbol** of the Lorentz group, because under boosts it does not change, i.e. for $\Lambda \in SO(1,3)$, we always have:

$$D_L(\Lambda)_a^c D_L(\Lambda)_b^d \epsilon_{cd} = \epsilon_{ab} \tag{46}$$

just as the metric does not change under Lorentz transformations in the fundamental, i.e. $\Lambda_\mu^\rho \Lambda_\nu^\sigma g_{\rho\sigma} = g_{\mu\nu}$. The close relation of ϵ to the metric g means that we can use the ϵ tensor to raise and dotted and undotted indices. In general, an **invariant symbol is a tensor which lives in the singlet representation**. To find invariant symbols for specific representations / tensors, one can look at the Clebsch-Gordan decomposition. Here are some common invariant symbols that one will find and the corresponding tensor decompositions; note the existence of an invariant symbol is a direct result of the decomposition containing the singlet $(0,0)$:

Symbol	Tensor decomposition
ϵ_{ab}	$(\frac{1}{2},0) \otimes (\frac{1}{2},0) = (0,0)_A \oplus (1,0)$
$\epsilon_{\dot{a}\dot{b}}$	$(0,\frac{1}{2}) \otimes (0,\frac{1}{2}) = (0,0)_A \oplus (0,1)$
$g_{\mu\nu}$	$(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) = (0,0)_S \oplus (0,1) \oplus (1,0) \oplus (1,1)$
$\epsilon_{\mu\nu\alpha\beta}$	$(\frac{1}{2},\frac{1}{2})^{\otimes 4} = (0,0)_A \oplus ...$
$\sigma^\mu_{a\dot{b}}$	$(\frac{1}{2},0) \otimes (0,\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) = (0,0) \oplus ...$

Table 2: Some common invariant symbols used in spinor analysis. Note that the representation $(\frac{1}{2},\frac{1}{2})$ is the fundamental vector representation of the Lorentz group. The subscripts A and S on the representations mean that they are either "antisymmetric" or "symmetric".

The fact that each of these tensors lives in the singlet representation implies that there is a generalized version of Eq. (46), and so we can use it to change indices with impunity because the symbol will not change under Lorentz transformation.

Let us consider a few examples. We know that the fundamental representation 4 is the same representation as $(\frac{1}{2},\frac{1}{2})$. So, we should be able to translate the components of a four vector V_μ into components of a tensor $V_{a\dot{a}}$. This is done with the invariant symbol $\sigma^\mu_{a\dot{b}}$, and simply contracting it with V_μ gives the desired components:

$$V_{a\dot{a}} = \sigma^\mu_{a\dot{a}} V_\mu \tag{47}$$

This is an explicit decomposition of the four-vector V_μ into components in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group.

Explicitly, the invariant symbol $\sigma^\mu_{a\dot{a}}$ is given by the σ^μ tensor, and has a counterpart in the $\overline{\sigma}$ tensor:

$$\sigma^\mu = \begin{pmatrix} 1 & \sigma^i \end{pmatrix} \qquad \overline{\sigma}^\mu = \begin{pmatrix} 1 & -\sigma^i \end{pmatrix} \tag{48}$$

$$\sigma^\mu_{a\dot{a}} = \sigma^\mu \qquad \sigma^{\mu\dot{a}a} = \overline{\sigma}^\mu \tag{49}$$

The generators $\mathcal{J}_{\mu\nu}$ in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ irreps can be covariantly expressed in terms of these two symbols:

$$(S_L^{\mu\nu})_a^{\ b} = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)_a^{\ b} \qquad (S_R^{\mu\nu})_{\dot{a}}^{\ \dot{b}} = -\frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)_{\dot{a}}^{\ \dot{b}} \tag{50}$$

As another example, any antisymmetric rank two tensor (i.e. like the field strength) $F_{\mu\nu}$ lives in $(1,0)\oplus(0,1)$. $(1,0)$ can be represented by a symmetric tensor with two left-handed Weyl (undotted) indices G_{ab}, and $(0,1)$ can likewise be represented with a symmetric tensor $G^\dagger_{\dot{a}\dot{b}}$, and we wish to express these tensors in terms of $F_{\mu\nu}$. We can use the generators S_L and S_R to map G_{ab} and $G_{\dot{a}\dot{b}}$ into antisymmetric Lorentz tensors:

$$G^{\mu\nu} = (S_L^{\mu\nu})^{ab} G_{ab} \qquad (G^\dagger)^{\mu\nu} = (S_R^{\mu\nu})^{\dot{a}\dot{b}} G^\dagger_{\dot{a}\dot{b}} \tag{51}$$

For G_{ab} and $G_{\dot{a}\dot{b}}$ in the $(1,0)$ and $(0,1)$ irreps, respectively, this correspondence will put symmetric undotted (dotted) 2-tensors in bijection with the (**anti**) **self-dual** Lorentz 2-tensors, which satisfy:

$$G^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma} \qquad (G^\dagger)^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} G^\dagger_{\rho\sigma} \tag{52}$$

Given $F_{\mu\nu}$, we can decompose it into a corresponding self-dual part $G_{\mu\nu}$ living in the $(1,0)$ irrep and an anti-self dual part $(G^\dagger)^{\mu\nu}$ living in $(0,1)$ by:

$$G^{\mu\nu} = \frac{1}{2}F^{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad (G^\dagger)^{\mu\nu} = \frac{1}{2}F^{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{53}$$

2.4 Discrete symmetries

Parity is a bit tricky for spin $\frac{1}{2}$ representations: since parity does not respect handedness, **parity does not exist for Weyl spinors**. To include parity in a theory, one must consider the Dirac representation of $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The parity operator in the bispinor representation is just equal to γ^0, since this connects $\psi_L \mapsto \psi_R$ and $\psi_R \mapsto \psi_L$:

$$D_{\text{bispinor}}(P) = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{54}$$

Charge conjugation C is another symmetry that must be discussed. There are a few ways to implement this symmetry. First, one can suppress all Dirac indices and use spinor notation. On a bispinor Ψ, charge conjugation acts as:

$$\Psi \xrightarrow{C} -i\gamma^2\Psi^* =: \Psi^{\mathrm{C}} \tag{55}$$

If we use our usual notation for a Dirac spinor, we can see what this rather opaque definition is telling us:

$$\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix} \mapsto \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \chi^* \\ \xi^{\dagger *} \end{pmatrix} = \begin{pmatrix} -i\sigma^2\xi^a \\ i\sigma^2\chi^* \end{pmatrix} = \begin{pmatrix} \epsilon_{ab}\xi^b \\ \epsilon_{\dot{a}\dot{b}}\chi^{\dagger\dot{b}} \end{pmatrix} \tag{56}$$

since $\epsilon^{ab} = i\sigma^2 = -\epsilon_{ab}$. The easiest way to remember this symmetry is through its action on spinor indices. From this point of view, **charge conjugation acts by interchanging the left/right handed pieces inside Ψ and raising / lowering the appropriate indices**:

$$\begin{pmatrix} \chi_a \\ \xi^{\dagger \dot{a}} \end{pmatrix} \xrightarrow{C} \begin{pmatrix} \xi_a \\ \chi^{\dagger \dot{a}} \end{pmatrix} \tag{57}$$

Note that as a matrix, $-i\gamma^2$ appears because it contains the ϵ tensors.

$$-i\gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_{ab} \\ \epsilon^{ab} & 0 \end{pmatrix} \tag{58}$$

Conventionally, charge conjugation can also be defined as $\Psi \mapsto C\overline{\Psi}^T$, where C is essentially the same matrix as $-\gamma^2$, just block-diagonal, as in this definition the upper and lower components of Ψ have already been flipped through Dirac conjugation. Here explicitly:

$$C = \begin{pmatrix} \epsilon_{ab} & 0 \\ 0 & \epsilon^{ab} \end{pmatrix} \tag{59}$$

3 The Poincaré Group

The Poincaré group is perhaps one of the most important groups in physics; it is the group of all isometries of Minkowski spacetime. Its representations are the setting of QFT, and at its core QFT is simply the best framework we have to combine the laws of quantum mechanics with the Poincaré symmetry. This group is denoted $ISO(3,1)$, and contains the Lorentz group as a subgroup; in addition to the Lorentz group, the Poincaré group also contains space-time translations. Its dimension is thus 10, with $SO(1,3)$ being identified as a 6 dimensional subgroup of $ISO(3,1)$.

The Poincaré group must therefore have 10 generators. We will soon redefine some of these generators to form the generators $\{J_i\}$, $\{K_i\}$ of the Lorentz group, but the Lorentz algebra, Equation 7, is not written covariantly. The generators of the Poincaré group are an antisymmetric tensor $J^{\mu\nu}$ (6 independent componenets) and a vector P^μ (4 independent components). These satisfy the Lie algebra:

$$\begin{aligned} i[J^{\mu\nu}, J^{\rho\sigma}] &= g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\sigma\mu} J^{\rho\nu} + g^{\sigma\nu} J^{\rho\mu} \\ i[P^\mu, J^{\rho\sigma}] &= g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho \\ [P^\mu, P^\sigma] &= 0 \end{aligned} \tag{60}$$

To show the Lorentz algebra is embedded in the Poincaré group's Lie algebra, one can define the three vectors:

$$\vec{J} := (J^{23}, J^{31}, J^{12}) \qquad \vec{K} := (J^{01}, J^{02}, J^{03}) \tag{61}$$

and with these definitions, $\vec{J}$ and $\vec{K}$ reproduce the algebra in Equation 7. The vector P^μ is the new addition to this group; this is the four-momentum that we all know and love. The spatial parts P^i generate spatial translations (so P^i is the 3-momentum) and the timelike part $H := P^0$ is the Hamiltonian and generates time translation. Written out in this form, in addition to the relations in Equation 7, the Poincaré algebra includes:

$$\begin{aligned}
[J_i, P_j] &= i\epsilon_{ijk}P_k \\
[K_i, P_j] &= -iH\delta_{ij} \\
[J_i, H] &= 0 \\
[P_i, H] &= 0 \\
[K_i, H] &= -iP_i
\end{aligned} \tag{62}$$

Particles in QFT live in the irreps of the Poincaré group, and as such they have labels which detail the irrep that they live in. Because the Poincaré group is not compact, its unitary irreps are not finite dimensional.

These labels are the mass and the spin of the particle. TODO.

3.1 The Little Group