Renormalization on the Lattice

Patrick Oare

1 Regularization-Independent Momentum Subtraction (RI-MOM)

The RI-MOM scheme (also known as Rome-Southampton) is a renormalization scheme well equipped to deal with the lattice, namely because we can calculate the relevant quantities that define the scheme easily on the lattice. The lattice spacing a provides us a with a natural UV regulator, and RI-MOM will tell us how to go from such a regulated result to a physical observable quantity.

The RI-MOM has a relatively simple renormalization condition. We will define it here for an arbitrary Green's function, for example a three point function. For a renormalization scale μ and working in a fixed gauge, **we define the amputated, renormalized Green's function at momentum $p^2 = -\mu^2$ to be equal to its tree level value**.

We will denote bare quantities as $a^{(0)}$, and renormalized quantities with a R subscript, when appropriate. Note that any quantities computed directly on the lattice are bare, and the entire point of using RI-MOM is to extract a sensible definition for these bare quantities.

In practice on the lattice, there are two things that we must compute. We will work with a specific example here for a given quark field $q(x)$. Suppose the operator we are trying to compute is $\mathcal{O}(z)$. We will renormalize the three point function:

$$G(p) = \frac{1}{V} \sum_{x,y,z} e^{ip\cdot(x-y)} \langle q(x)\mathcal{O}(z)\overline{q}(y)\rangle \tag{1}$$

i.e. we are projecting the source and sink to a definite momentum and projecting the operator $\mathcal{O}(z)$ to zero momentum.

We also will need to compute the momentum projected propagator:

$$S(p) = \frac{1}{V} \sum_{x,y} e^{ip\cdot(x-y)} S(x, y) \tag{2}$$

where $S^{ab}_{ij}(x, y) = \langle q^a_i(x)\bar{q}^b_j(y)\rangle$ is the standard propagator. On the lattice, the two objects that we must compute directly are $S(p)$ and $G(p)$, and everything else follows once we have these quantities.

To explain the method, our goal is to compute the operator renormalization:

$$\mathcal{O}_R(\mu) = \mathcal{Z}(\mu)\, \mathcal{O}^{(0)} \tag{3}$$

where $\mathcal{O}_R(\mu)$ is our renormalized operator, $\mathcal{Z}(\mu)$ is the renormalization coefficient of interest, and $\mathcal{O}^{(0)}$ is the lattice (bare) operator. We will also assume the quark fields have been renormalized by some quark field renormalization $\mathcal{Z}_q$ (note this is the opposite convention studied in many QFT classes):

$$q_R(\mu) = \sqrt{\mathcal{Z}_q}\, q^{(0)} \tag{4}$$

There is an analytical expression for $\mathcal{Z}_q$ on the lattice in the RI-MOM scheme, and it has been determined to be:

$$\mathcal{Z}_q(p)|_{p^2=-\mu_R^2} = i\frac{1}{12\tilde{p}^2} tr\left\{S^{-1}(\tilde{p})\tilde{\not{p}}\right\}\Big|_{p^2=\mu^2} = \left[\frac{tr\left\{i\sum_{\nu=1}^4 \gamma_\nu \sin(ap_\nu)aS(p)^{-1}\right\}}{12\sum_{\nu=1}^4 \sin^2(ap_\nu)}\right]_{p^2=\mu^2} \tag{5}$$

The twelve on the bottom is a normalization $12 = 3 \times 4$ for the number of color and number of spin indices, which we will see in many of the expressions. Here we are using the discretized momentum $\tilde{p}_\mu = \frac{1}{a}\sin(ap_\mu)$ which replaces the continuum momentum on the lattice. Note that many papers in the literature use the negative of this expression. Because our phase convention is opposite the $e^{-ip(x-y)}$ convention, the sign of our momenta are opposite to this other convention, and so we pull an extra negative sign into Equation 5

Let $\Gamma(p)$ be the **amputated three point function**, bare or renormalized. We can relate Γ to the other quantities we have already computed by using the inverse propagator to manually cut the legs off the full three point function:

$$\Gamma(p) = S(p)^{-1}G(p)S(p)^{-1} \tag{6}$$

We will denote the tree level version of this by Γ_B, where the B subscript stands for "Born".

Using our quantities already computed on the lattice, we can compute the bare $\Gamma^{(0)}(p)$ directly in terms of the renormalized quantities. In the continuum limit, the renormalized Green's function will be:

$$G_R(p;\mu) = \int d^4x\, d^4y\, d^4z\, e^{ip\cdot(x-y)}\langle q_R(x;\mu)\mathcal{O}_R(z;\mu)\overline{q}_R(y;\mu)\rangle \tag{7}$$

$$= \mathcal{Z}_q(\mu)\mathcal{Z}(\mu)\int d^4x\, d^4y\, d^4z\, e^{ip\cdot(x-y)}\langle q^{(0)}(x)\mathcal{O}^{(0)}(z)q^{(0)}(y)\rangle \tag{8}$$

$$= \mathcal{Z}_q(\mu)\mathcal{Z}(\mu)G^{(0)}(p) \tag{9}$$

Similarly, the bare and renormalized propagators are related as:

$$S_R(p;\mu) = \int d^4x d^4y\, e^{ip\cdot(x-y)}\langle q_R(x;\mu)\overline{q}_R(y;\mu)\rangle = \mathcal{Z}_q(\mu)S^{(0)}(p) \tag{10}$$

These relation translates immediately to the amputated Green's function $\Gamma(p)$, as $S_R^{-1}(p;\mu) = \mathcal{Z}_q^{-1}(\mu)(S^{(0)})^{-1}$ we see that:

$$\Gamma_R(p;\mu) = \mathcal{Z}_q(\mu)^{-1}\mathcal{Z}(\mu)\Gamma^{(0)}(p) \tag{11}$$

We are now in a position to apply the renormalization condition. We must equate the renormalized, amputated Green's function to the tree level Green's function $\Gamma_B(p)$. This gives us:

$$\mathcal{Z}_q(\mu)^{-1}\mathcal{Z}(\mu)\Gamma(p) = \Gamma_B(p) \tag{12}$$

Dividing by a conventional factor of 12 as a normalization and inverting, we can clean this expression up into a simple equation for $\mathcal{Z}(\mu)$:

$$\mathcal{Z}(p^2 = -\mu^2) = \left[\frac{12\mathcal{Z}_q(p)}{tr\left\{\Gamma(p)\Gamma_B(p)^{-1}\right\}}\right]_{p^2=\mu^2} \tag{13}$$

2 Isospin

We are interested here in computing matrix elements of the operator:

$$\mathcal{O}(z) = \mathcal{O}_u(z) - \mathcal{O}_d(z) \tag{14}$$

where the quark operators $\mathcal{O}_q$ are given by

$$\mathcal{O}_q(z) = \frac{1}{\sqrt{2}}(\mathcal{T}^q_{33}(z) - \mathcal{T}^q_{44}(z)) \tag{15}$$

and the irreducible tensor operators $\mathcal{T}^q_{\mu\nu}$ are defined as

$$\mathcal{T}^q_{\mu\nu} = \overline{q}(z)\,\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}\,q(z) \tag{16}$$

with $\overleftrightarrow{D} = \vec{D} - \overleftarrow{D}$ the antisymmetrized covariant derivative. Note we define the symmetric and traceless component of a tensor to be:

$$a_{\{\mu}b_{\nu\}} = \frac{1}{2}(a_\mu b_\nu + a_\nu b_\mu) - \frac{1}{4}a_\alpha b^\alpha g_{\mu\nu} \tag{17}$$

We are inserting the operator $\mathcal{O}_q$ with momentum $\vec{p} = 0$, i.e. dealing with $\sum_z \mathcal{O}_q(z) = \mathcal{O}_q(\vec{p} = 0)$. For this current insertion, we can read off the Born term (the tree level matrix element) by replacing the covariant derivative $\overleftrightarrow{D}$ with the sink momentum p[1]. This gives us:

$$\Gamma_B(p) = i\sqrt{2}(\tilde{p}_3\gamma_3 - \tilde{p}_4\gamma_4) \tag{18}$$

where $\tilde{p}$ is taken to be the lattice momentum $\tilde{p}_\mu = 2\sin(\pi k_\mu/L_\mu)$.

However, this is not completely accurate. There is mixing from other tensor structures, and we must account for that as well. There are two ways to deal with this.

The first method we can check is to solve for the Born term directly on the lattice by solving the free theory (initialized with $U^{ab}_\mu(x) = \delta^{ab}$) for Γ, the amputated Green's function, as the Born term is defined to be $\Gamma(p)$ in the non-interacting theory. To implement this, we perform a color-averaged trace $\frac{1}{3}tr_C\{S\}$ of the propagator to strip off the color indices and leave ourselves with a Dirac matrix, and then use the corresponding Dirac matrix as the Born term. We note that this should match closely to our original construction of the Born term with additional $O(a)$ artifacts, since the other tensor structure not accounted for in Equation 18 arises from the $O(4) \to H(4)$ symmetry breaking.

The second method redefines the renormalization condition in Equation 13 to account for the mixing directly. We expand the amputated Green's function $\Gamma(p)$ as a linear combination of the two tree level structures:

$$i\Lambda^1_{\mu\nu} := \frac{1}{2}(p_\mu\gamma_\nu + p_\nu\gamma_\mu) - \frac{1}{4}\not{p}\delta_{\mu\nu} \tag{19}$$

$$i\Lambda^2_{\mu\nu} := \frac{p_\mu p_\nu}{p^2}\not{p} - \frac{1}{4}\not{p}\delta_{\mu\nu} \tag{20}$$

[1] In a Gockeler's analysis of a similar operator, the Born term for $\mathcal{O}_{\mu\nu} = \frac{1}{2}\overline{u}(\gamma_\mu\overleftrightarrow{D}_\nu + \overleftrightarrow{D}_\mu\gamma_\nu)d$ is taken to be $\Gamma_B(p) = i(\gamma_\mu p_\nu + p_\mu\gamma_\nu)$

where we use the lattice momentum for p. If we let $\Gamma^D(p) := \frac{1}{3}tr_C\{\Gamma(p)\}$ be the Dirac structure of $\Gamma(p)$, then this expansion amounts to imposing the renormalization condition:

$$\Gamma^D_{\mu\nu}(p)\Big|_{p^2=\mu^2} = \Pi^1(p^2)\Lambda^1_{\mu\nu} + \Pi^2(p^2)\Lambda^2_{\mu\nu} \tag{21}$$

where $\Pi^1(p^2)$ and $\Pi^2(p^2)$ are coefficients which are related to the renormalization coefficients by:

$$\Pi^i(p^2 = \mu^2) = \frac{Z_q}{Z_i(p^2 = \mu^2)} \tag{22}$$

and so knowledge of the Π^i coefficients will allow us to extract the renormalization coefficient $Z_i(p^2 = \mu^2)$. To solve for this, we construct a linear functional on the space of Dirac and Lorentz matrices by defining a form $\langle\cdot,\cdot\rangle$ which is defined by

$$\langle\lambda^1, \lambda^2\rangle := \sum_{i\in\mathfrak{R}} tr_D\left\{\lambda^1_i\lambda^2_i\right\} \tag{23}$$

where λ^1, λ^2 are objects with two Dirac and two Lorentz indices. Here, $\mathfrak{R}$ denotes the irrep of $H(4)$ that we are working in. In our case, we are considering $\mathfrak{R} = \tau_1^{(3)}$, one of the 3 dimensional irreps of $H(4)$, and so we sum over the normalized basis elements of the irrep:

$$\tau_1^{(3)} = span\Bigg\{\frac{1}{\sqrt{2}}\left(\mathcal{O}_{\{33\}} - \mathcal{O}_{\{44\}}\right), \tag{24}$$

$$\frac{1}{\sqrt{2}}\left(\mathcal{O}_{\{11\}} - \mathcal{O}_{\{22\}}\right), \tag{25}$$

$$\frac{1}{2}\left(\mathcal{O}_{\{11\}} + \mathcal{O}_{\{22\}} - \mathcal{O}_{\{33\}} - \mathcal{O}_{\{44\}}\right)\Bigg\} \tag{26}$$

We apply the functionals $\langle\Lambda^1,\cdot\rangle$ and $\langle\Lambda^2,\cdot\rangle$ to Equation 21 to obtain the following system of equations which we can use to solve for the coefficients Π^1 and Π^2:

$$\mathcal{A}^{ab}\Pi^b = \begin{pmatrix}\langle\Lambda^1, \Gamma(p)\rangle\\ \langle\Lambda^2, \Gamma(p)\rangle\end{pmatrix} \tag{27}$$

where $\mathcal{A}^{ab}$ is the matrix of inner products:

$$\mathcal{A}^{ab} := \langle\Lambda^a, \Lambda^b\rangle \tag{28}$$

where a, b range over 1 and 2. Using the explicit form for Λ^1 and Λ^2, the matrix $\mathcal{A}^{ab}$ can be computed directly. For example:

$$\mathcal{A}^{12} = \langle\Lambda^1, \Lambda^2\rangle \tag{29}$$

$$= \frac{1}{2}tr\Big\{(\Lambda^1_{33} - \Lambda^1_{44})(\Lambda^2_{33} - \Lambda^2_{44})\Big\} + \frac{1}{2}tr\Big\{(\Lambda^1_{11} - \Lambda^1_{22})(\Lambda^2_{11} - \Lambda^2_{22})\Big\} \tag{30}$$

$$+ \frac{1}{4}tr\Big\{(\Lambda^1_{11} + \Lambda^1_{22} - \Lambda^1_{33} - \Lambda^1_{44})(\Lambda^2_{11} + \Lambda^2_{22} - \Lambda^2_{33} - \Lambda^2_{44})\Big\} \tag{31}$$

and these traces can be evaluated using the identities $tr\{\gamma_\mu\gamma_\nu\} = 4\delta_{\mu\nu}$, $(\not{k})^2 = k^2$, and $tr\{1\} = 4$. The result is:

$$\mathcal{A}^{ab}(p) = \begin{pmatrix} 3\tilde{p}^2 & \frac{1}{\tilde{p}^2}\left(-3\tilde{p}^{[4]} + 2\sum_{i<j}\tilde{p}_i^2\tilde{p}_j^2\right)\\ \frac{1}{\tilde{p}^2}\left(-3\tilde{p}^{[4]} + 2\sum_{i<j}\tilde{p}_i^2\tilde{p}_j^2\right) & \frac{1}{\tilde{p}^2}\left(-3\tilde{p}^{[4]} + 2\sum_{i<j}\tilde{p}_i^2\tilde{p}_j^2\right)\end{pmatrix} \tag{32}$$

where $p^{[4]}$ is the hypercubic invariant $\sum_i p_i^4$.

We will focus our analysis on the tensor operator $\mathcal{T}^q_{\mu\mu}$ (note there is no sum on μ here), and note that $\mathcal{O}(z)$ can be obtained through linearity. We may write:

$$\sum_z \mathcal{T}^q_{\mu\mu}(z) = \sum_{z,z'} \overline{q}(z)\, J_\mu(z,z')\, q(z') \tag{33}$$

Plugging in the definition of the derivatives:

$$\vec{D}\psi(z) = \frac{1}{2}\Big(U_\mu(z)\psi(z+\hat{\mu}) - U_\mu(n-\hat{\mu})^\dagger \psi(z-\hat{\mu})\Big) \tag{34}$$

$$\overline{\psi}(z)\overleftarrow{D} = \frac{1}{2}\Big(\overline{\psi}(z+\hat{\mu})U_\mu(z)^\dagger - \overline{\psi}(z-\hat{\mu})U_\mu(z-\hat{\mu})\Big) \tag{35}$$

we find the current $J_\mu(z,z')$ is:

$$J_\mu(z,z') = \Big[U_\mu(z)\delta_{z+\hat{\mu},z'} - U_\mu(z')^\dagger \delta_{z-\hat{\mu},z'}\Big]\,\gamma_\mu \tag{36}$$

We may now use this expansion to compute the three point function for the operator $\mathcal{T}_\mu = \mathcal{T}^u_{\mu\mu} - \mathcal{T}^d_{\mu\mu}$ (we can simply take $\mathcal{T}_3 - \mathcal{T}_4$ to get the operator of interest in Equation 14). Using Equation 33, we write

$$\sum_z \mathcal{T}_\mu(z) = \sum_{z,z'} \big[\overline{u}(z)\, J_\mu(z,z')\, u(z') - \overline{d}(z)\, J_\mu(z,z')\, d(z')\big] \tag{37}$$

Plugging this into Equation 1, we find that we can expand the total up quark Green's function (here α, β are Dirac indices) as:

$$G^{\alpha\beta}(p) = \frac{1}{\sqrt{2}}\Big(G_3^{\alpha\beta}(p) - G_4^{\alpha\beta}(p)\Big) \tag{38}$$

where:

$$G_\mu^{\alpha\beta}(p) = \frac{1}{V}\sum_{x,y,z} e^{-ip(x-y)}\langle u^\alpha(x)\mathcal{T}_\mu(z)\overline{u}^\beta(y)\rangle \tag{39}$$

$$= \frac{1}{V}\sum_{x,y,z,z'} e^{-ip(x-y)}\Big[\langle u^\alpha(x)\overline{u}^\sigma(z)J_\mu^{\sigma\rho}(z,z')u(z')^\rho\overline{u}^\beta(y)\rangle - \langle u^\alpha(x)\overline{d}^\sigma(z)J_\mu^{\sigma\rho}(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle\Big] \tag{40}$$

Now we perform all possible Wick contractions on the matrix elements to write them as propagators:

$$\begin{aligned}\langle u^\alpha(x)\overline{u}^\sigma(z)J_\mu^{\sigma\rho}(z,z')u^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{u}Ju\overline{u}\rangle + \langle u\overline{u}Ju\overline{u}\rangle \\ &= S^{\alpha\sigma}(x,z)J_\mu^{\sigma\rho}(z,z')S^{\rho\beta}(z',y) + (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)\end{aligned} \tag{41}$$

$$\begin{aligned}\langle u^\alpha(x)\overline{d}^\sigma(z)J_\mu^{\sigma\rho}(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{d}Jd\overline{u}\rangle \\ &= (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)\end{aligned} \tag{42}$$

where the factors of (-1) come from rearranging the contraction so that the contracted pieces are of the form $\langle u\overline{u}\rangle$. The vacuum pieces cancel because the up and down quark propagators are degenerate, so the final result is very clean:

$$G_\mu(p) = \frac{1}{V}\sum_{x,y,z,z'} e^{ip(x-y)}S(x,z)J_\mu(z,z')S(z',y) \tag{43}$$

This is our central equation, but note that computing this directly on the lattice would involve computing the two point propagator $S(x,y)$ at each two points on the lattice. This is much too computationally intensive, so we must resort to other techniques to accomplish this.

There are two primary ways to compute this on the lattice. We can compute this directly using momentum sources, or we can use the sequential source technique. Momentum sources work specifically for Equation 43, but produce a significantly better signal on a small number of configurations. On the other hand, sequential source is much more general, but produces more noise. We will discuss each method below.

2.1 Momentum sources

Observe that we can rewrite Equation 43 as:

$$\begin{aligned}
G_\mu(p) &= \frac{1}{V}\sum_{x,y,z,z'} e^{ipx}S(x,z)J_\mu(z,z')e^{-ipy}S(z',y) \\
&= \frac{1}{V}\sum_{z,z'}\gamma_5\left(\sum_x S(z,x)e^{-ipx}\right)^\dagger\gamma_5 J_\mu(z,z')\left(\sum_y S(z',y)e^{-ipy}\right) \\
&= \frac{1}{V}\sum_{z,z'}\gamma_5\tilde{S}_p(z)^\dagger\gamma_5 J_\mu(z,z')\tilde{S}_p(z')
\end{aligned} \tag{44}$$

where we have defined $\tilde{S}_p(z)$ as:

$$\tilde{S}_p(z) = \sum_x S(z,x)e^{-ipx} \tag{45}$$

The advantage of casting the equation in this form is that we can solve for $\tilde{S}_p(z)$ directly by inverting the Dirac equation with a momentum source, i.e. we have:

$$\sum_z D(x,z)\tilde{S}_p(z) = e^{-ipx} \tag{46}$$

where $D(x,z)$ is the Dirac operator. This means that upon solving for $\tilde{S}_p(z)$ and plugging this into Equation 44, we can solve directly for $G_\mu(p)$.

We can also use the propagator we get from the momentum source inversion to directly compute the propagator in Equation 2 as follows:

$$S(p) = \frac{1}{V}\sum_{x,y} e^{ip\cdot(x-y)}S(x,y) = \frac{1}{V}\sum_x e^{ip\cdot x}\tilde{S}_p(x) \tag{47}$$

This is an exact equation and it does not rely on translational invariance in the infinite statistics limit. Therefore, this method will give much better signal and can be run efficiently on a small number of configurations. The downside to this is that we require a propagator inversion for each choice of sink momentum. To compute $G(p)$ for a large number of sink momenta, as we need to do to extrapolate $\mathcal{Z}(\mu)$ in the continuum limit, a propagator inversion at each sink momenta is not feasible. We must instead choose the sink momentum wisely to be able to extract the discretization artifacts and extrapolate $\mathcal{Z}(\mu)$ to the continuum, which we will describe in Section 4.

2.2 Sequential source method

In practice we will use the sequential source method, which if implemented correctly does not force us to invert a propagator at every sink momenta. This technique is also much more general than the one previously described, but it suffers from more noise because it relies on the translational invariance of the lattice, which only exists in the infinite statistics limit. The idea of the sequential source method is that if we have an equation involving the full propagator $S(x, y)$, we can invert a source which depends on the propagator $S(x)$. For example, in this problem we wish to evaluate Equation 43, but we cannot simply evaluate $S(x, y)$ for every x and y. To get around this, consider using a source

$$b(z) = \sum_{z'} J_\mu(z, z')S(z', 0) \tag{48}$$

to invert the Dirac equation, which will solve for $M(x)$ in this equation:

$$\sum_x D(z, x)M(x) = b(z) \tag{49}$$

where $D(x, z)$ is the Dirac operator. Upon inversion, using that $\sum_z S(y, z)D(z, x) = \delta(y - x)$, we can move the Dirac operator to the other side as $D^{-1}(y, z) = S(y, z)$ and obtain:

$$M(x) = \sum_z D^{-1}(x, z)b(z) = \sum_{z,z'} S(x, z)J_\mu(z, z')S(z', 0) \tag{50}$$

Note that we have summed the full propagator $S(x, z)$ for the price of a single inversion of the source $b(z)$. We can then reconstruct Equations 2 and 43 for the momentum-projected Green's function and propagator in the infinite statistics limit when translational invariance is restored:

$$G_\mu(p) \approx \frac{1}{V}\sum_x e^{ipx}M(x) = \frac{1}{V}\sum_{x,z,z'} e^{ipx}S(x, z)J_\mu(z, z')S(z', 0) \tag{51}$$

$$S(p) \approx \frac{1}{V}\sum_x e^{ipx}S(x, 0) \tag{52}$$

We can improve the performance of this by randomly sampling N lattice points to approximate the sums over y in our equations for $S(p)$ and $G(p)$. If we sample the points $\{y_i\}_{j=1}^{N}$, then we can make the replacement $\sum_y \mapsto \frac{V}{N}\sum_j$, and so we have:

$$G_\mu(p) \approx \frac{1}{N}\sum_{x,z,z'}\sum_j e^{ip(x-y_j)}S(x, z)J_\mu(z, z')S(z', y_j) \tag{53}$$

$$S(p) \approx \frac{1}{N}\sum_x\sum_j e^{ip\cdot(x-y_j)}S(x, y_j) \tag{54}$$

Since we are interested in renormalizing the operators in each irrep, we must take the operators we are measuring to be:

$$\mathcal{O}_1 = \frac{1}{\sqrt{2}}\left(\mathcal{T}_{33} - T_{44}\right) \tag{55}$$

$$\mathcal{O}_2 = \frac{1}{\sqrt{2}}\left(\mathcal{T}_{11} - T_{22}\right) \tag{56}$$

$$\mathcal{O}_3 = \frac{1}{2}\left(\mathcal{T}_{11} + \mathcal{T}_{22} - \mathcal{T}_{33} - T_{44}\right) \tag{57}$$

which give us currents:

$$\mathcal{J}_1 = \frac{1}{\sqrt{2}}(J_3 - J_4) \tag{58}$$

$$\mathcal{J}_2 = \frac{1}{\sqrt{2}}(J_1 - J_2) \tag{59}$$

$$\mathcal{J}_3 = \frac{1}{2}(J_1 + J_2 - J_3 - J_4) \tag{60}$$

and we must compute the 3 propagators using $\mathcal{J}_i$ QLUA.

To check an implementation of this method, we can invert a sequential propagator which depends on momentum. If we replace $S(z', 0)$ in Equation 48 and invert, we find that:

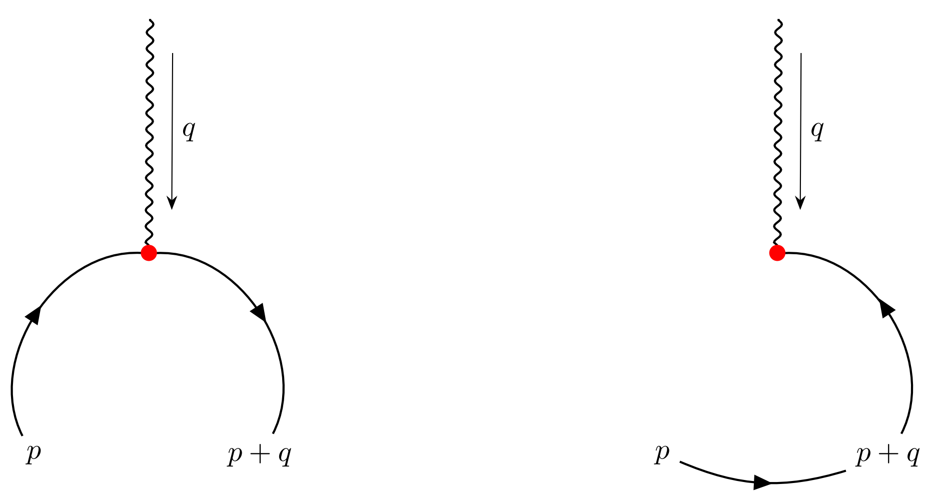
$$b_\mu^{(p)}(z) = \sum_{z'} J_\mu(z, z')\tilde{S}_p(z') = \sum_{z',y} e^{-ipy} J_\mu(z, z') S(z', y) \tag{61}$$

$$M_\mu^{(p)}(x) = \sum_{z} S(x, z) b_\mu^{(p)}(z) = \sum_{z,z',y} e^{-ipy} S(x, z) J_\mu(z, z') S(z', y) \tag{62}$$

When we momentum project to find the Green's function $G(p)$, this no longer relies on translation invariance and should match our result with the momentum inversion exactly:

$$G_\mu(p) = \sum_{x} e^{ipx} M_\mu^{(p)}(x) = \sum_{x,y,z,z'} e^{ip(x-y)} S(x, z) J_\mu(z, z') S(z', y) \tag{63}$$

In our case with a large amount of sink momenta, this method is much more robust than inverting a momentum source because we one inversion can give us $G(p)$ at every value of the sink momentum. We will call this construction going **through the operator**, because the inversion in Equation 50 projects the current insertion onto $q = 0$ momentum. If we had been interested in projecting the operator onto different momentum values, then we would need to use a new sequential source (modify Equation 48) for each value of the operator momentum. Pictorially, we are inverting at the operator momentum, then tying up at the sink momentum. On the other hand, we reverse the direction of inversion and invert our propagator at each sink momentum first, then tie up the line at the operator. This method is called going **through the sink**. We can represent these different methods below, where in our case $q = 0$.



(a) Inversion through the operator

(b) Inversion through the sink

In this problem inversion through the sink would require too many propagator inversions like in the previous momentum source method, and it would also be noisy like inversion through the operator. As such, there is no reason to consider it, and I included it here mainly for generality.

3 Matching to a continuum scheme

We must now discuss how to convert between schemes, because typically we want our results to be in a common scheme used by many other researchers. For our case, we will convert the renormalization constant $Z_{RI-MOM}(p)$ which we have calculated in the previous sections to the $\overline{MS}$ scheme.

There are a few cases we must consider. Most papers explicitly compute a

Recall that we can express the β function of a theory in a scheme $\mathcal{S}$ using a perturbative expansion in the coupling g or in $\alpha = \frac{g^2}{4\pi}$ as follows:

$$\beta_{\mathcal{S}}(g) = -\beta_0 \frac{g_{\mathcal{S}}^3}{16\pi^2} - \beta_1 \frac{g_{\mathcal{S}}^5}{(16\pi^2)^2} - \beta_2^{\mathcal{S}} \frac{g_{\mathcal{S}}^7}{(16\pi^2)^3} - ... \tag{64}$$

where the β_j are coefficients. Note that β_0 and β_1 are independent of scheme and given by:

$$\beta_0 = \frac{11}{3} - N_f \tag{65}$$

$$\beta_1 = \tag{66}$$

4 Hypercubic Artifacts

When we compute observables at a finite lattice spacing a, we suffer **discretization artifacts** which are relics of the explicit symmetry breaking $SO(1,3) \to H(4)$ suffered by putting the theory on a lattice. Namely, because we have less symmetry, there are more invariant quantities of p^μ in a lattice theory than in the continuum. In the continuum, the basic invariant that we can create is $p^2 = p_\mu p^\mu$.

There are two basic types of discretization artifacts which will show up: artifacts from the breaking of Lorentz invariance $O(4) \to H(4)$, and artifacts from running. The symmetry breaking artifacts will appear in the data as “fans”. When these are removed, the data will appear to have a smooth structure, at which point one can solve for the running artifacts by expanding in all possible terms consistent with symmetry.

To fit the artifacts from symmetry breaking, we may use the **$\mathbf{p^{[2n]}}$ extrapolation method**. This amounts to expanding the artifacts in a Taylor series in hypercubic invariants.

On the lattice, we can find other invariants because the orbits of $H(4)$ are strictly smaller than the orbits of $O(4)$, the Euclidean isometry group in $d = 4$. For example, the vectors $(2,0,0,0)$ and $(1,1,1,1)$ have the same value of $p^2 = 4$, yet there is no element $g \in H(4)$ such that $g \cdot (1,1,1,1) = (2,0,0,0)$, i.e. they cannot be rotated into one another by hypercubic symmetry. This is because we can define *other hypercubic invariants than just* p^2. The functions:

$$p^{[2n]} := \sum_\mu p^{2n} \tag{67}$$

for $n \in \mathbb{N}$ are also invariants, and these dictate the orbits of momenta under $H(4)$. Since $(1,1,1,1)$ has $p^{[4]} = 4$ and $(2,0,0,0)$ has $p^{[4]} = 16$, we can conclude they are distinct momenta under hypercubic symmetry and thus cannot live in the same orbit of $H(4)$.

Any function which is invariant under the action of $H(4)$ must be a function of these hypercubic invariants, much like how in the continuum any function which was invariant under Lorentz symmetry was a function of Lorentz scalars like p^2 or $\not{p}$. Because we are computing out quantities on the lattice which only has $H(4)$ symmetry, these extra terms like $p^{[4]}$ can come into play when form factors or renormalization constants are computed, and this extrapolation method will account for these.

Another source of error that we must consider when performing calculations on the lattice is that lattice momenta is quantized. The possible values that the momenta can take are:

$$p_\mu = \frac{2\pi}{aL_\mu} k_\mu \tag{68}$$

where $k_\mu \in \mathbb{Z}$ and L_μ is the size of the lattice in direction μ. In the lattice action, the momentum is modified to become:

$$\tilde{p}_\mu = \frac{1}{a} \sin(ap_\mu) \tag{69}$$

and in the small a limit, notice that this reduces to the standard momentum values p_μ. As a result, our renormalization coefficient computed on the lattice will be a function of $\tilde{p}_\mu$, *not* a function of p_μ. We wish to account for this and for our final result to be a function of p_μ, so we must take this into account when fitting the hypercubic artifacts.

TODO finish this section