HOMOLOGY

PATRICK OARE

Like the fundamental group, homology is an important algebraic invariant of topological spaces which can be used to describe the properties of such spaces. Although it is more complicated to define, it provides a powerful tool to use to classify spaces. The essence of homology is to define a sequence of functors $H_n : Top \to Ab$ which are invariant up to homotopy. A particular advantage of $H_n(\cdot)$ over homotopy groups $\pi_n(\cdot)$ is that the homology groups are abelian, and thus their structure is much easier to understand than their non-abelian counterparts.

We will begin by studying simplicial homology, which is a particular example of a homology theory. Intuitively, the simplicial homology groups of a space will classify how many "n-dimensional holes" the space has. Eventually, we will generalize the notion of homology to an arbitrary chain complex and examine chain complex structures that may arise as we consider different examples of spaces.

1. Simplicial Homology

We begin with simplicial homology, which is a particular way to construct homology groups on a space. We will aim to study a space X based on maps from simpler spaces (the n-simplices) into X.

Definition 1.1 (Standard n-simplex)**.** For $n > 0$, the **standard n-simplex** Δ^n is the convex hull of the standard basis $\{e_0, ..., e_n\}$ of $\mathbb{R}^{n+1}$, i.e. we have:

$$\Delta^n := \left\{ \sum_{k=0}^{n} t_k e_k : \sum_{k=0}^{n} t_k = 1, t_k \geq 0 \right\} \tag{1}$$

The numbers $(t_0, ..., t_k)$ are called **barycentric coordinates**.

The standard n-simplex is an extraordinary simple space, and just looks like a hyper-triangle embedded into $\mathbb{R}^{n+1}$. We can describe the n-simplex as an ordered pair $[v_0, ..., v_n]$, where each $v_i \in \{0, ..., n\}$ and none are repeated. This assigns each simplex an implicit orientation, and we can use this to manipulate such simplices. For each n, we have inclusion maps:

$$d^i : \Delta^{n-1} \to \Delta^n, \;\; 0 \leq i \leq n \tag{2}$$

where we map $[v_0, ..., v_{n-1}]$ to $[v_0, ..., \hat{v}_i, ..., v_n]$, where the $\hat{v}_i$ denotes that we omit the vertex v_i.

Definition 1.2 (Singular n-simplex)**.** Let X be a topological space. Then a **singular n-simplex** is a map:

$$\sigma : \Delta^n \to X \tag{3}$$

We define $Sin_n(X)$ to be the set of all singular n-simplices in X.

Note that the face maps induce a canonical map between the singular n-simplices on X through the following composition of maps, which will call $d_i : Sin_n(X) \to Sin_{n-1}(X)$, $\sigma \mapsto \sigma \circ d^i$:

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "d^i"] \arrow[dr, "d_i\sigma"'] & \Delta^n \arrow[d, "\sigma"] \\ & X \end{tikzcd} \tag{4}$$

For $i \leq j$, the maps d_i satisfy (and the face maps d^i satisfy the same identity):

$$d_i d_j = d_{j+1} d_i \tag{5}$$

We wish to consider a variant of $Sin_n(X)$ which allows us to add and subtract simplices in a natural way. This will allow us to consider boundaries of simplices, which will be intimately connected to homology. For example, consider a 1-simplex which is a closed loop, i.e. $\gamma : [0,1] = \Delta^1 \to X$. We wish to be able to differentiate this loop from an open loop. We can almost do this with the definitions we have already made: notice that $d_0(\gamma) = \gamma(1) = \gamma(0) = d_1(\gamma)$. However, we need a way to take differences to make this precise. As such, we will consider the free abelian group generated by $Sin_n(X)$.

Definition 1.3 (Singular n-chain)**.** Let $S_n(X)$ be the free abelian group generated by $Sin_n(X)$:

$$S_n(X) := \mathbb{Z}Sin_n(X) \tag{6}$$

We call an element of $S_n(X)$ a **singular n-chain**, and we may write each n-chain as:

$$\sum_{k=1}^{n} a_k \sigma_k \tag{7}$$

for $\sigma_k \in Sin_n(X)$.

Definition 1.4 (Boundary operator)**.** We define the **boundary operator** (also called a differential) $d : Sin_n(X) \to S_{n-1}(X)$ by:

$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i d_i \sigma \tag{8}$$

This extends uniquely to a homomorphism:

$$d : S_n(X) \to S_{n-1}(X) \tag{9}$$

We will use this boundary map extensively, as it allows us to go between simplices of different dimensions. Intuitively, the boundary map just gives you the oriented face of the n-simplex; for example, the boundary of Δ^2 is just an oriented (not filled in) triangle. Simplices which are killed by the differential are "boundaries" of closed regions, in a sense which we will make precise.

Definition 1.5 (n-cycle)**.** An n-chain c in X is a **n-cycle** if $dc = 0$. We denote the set of all n-cycles by:

$$Z_n(X) := ker(d : S_n(X) \to S_{n-1}(X) \tag{10}$$

Definition 1.6 (n-boundary)**.** An n-chain b in X is a **n-boundary** if $b \in im(d)$. We denote the set of all n-boundaries by:

$$B_n(X) := im(d : S_{n+1}(X) \to S_n(X)) \tag{11}$$

Theorem 1.1. The boundary operator satisfies:

$$d^2 = 0 \tag{12}$$

This implies that $B_n(X) \subseteq Z_n(X)$, so every boundary is a cycle.

Definition 1.7 (Graded abelian group)**.** A **graded abelian group** is a sequence of abelian groups indexed by $\mathbb{Z}$. A **chain complex** is a graded abelian group $\{A_n\}_{n\in\mathbb{Z}}$ together with homomorphisms $d : A_n \to A_{n-1}$ such that $d^2 \equiv 0$. We will draw a chain complex like so:

$$... \xrightarrow{d} A_{n+1} \xrightarrow{d} A_n \xrightarrow{d} A_{n-1} \xrightarrow{d} ... \tag{13}$$

An n-cycle in $Z_n(X)$ is closed in the way that a closed loop is closed– it has no edges and all its faces connect with itself. The distinction between cycles and boundaries will give us homology groups. A nice way to visualize the difference is to consider an annulus in $\mathbb{R}^2$, $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in [1, 3]\}$. A circle wrapping the annulus $x^2 + y^2 = 2$ is a cycle because it is a closed path; the boundary operator sends it to 0 since it links back to itself. However, it is **not** a boundary. It could almost be the boundary of $U := \{(x, y) : x^2 + y^2 \leq 2\} \cap A$, but this is not a simplex. U is 2-dimensional and resembles the simplex Δ^2, but unfortunately has a hole in it.

In a manner such as this, homology will tell us about the holes that a space has by considering the quotient of the n-cycles by the n-boundaries.

Definition 1.8. For a topological space X, the nth **singular homology group** of X is:

$$H_n(X) := Z_n(X)/B_n(X) \tag{14}$$

Homology can also be defined in the same way for an arbitrary chain complex, and often once we have a chain complex we will forget about the underlying space it comes from. Note that homology groups $H_n(X)$ are always abelian, since $Z_n(X)$ is an abelian group.

Definition 1.9 (Semi-simplicial set)**.** A collection of sets K_n for $n \geq 0$ together with maps $d_i : K_n \to K_{n-1}$ satisfying $d_i d_j = d_{j+1} d_i$ for $i \leq j$ is called a **semi-simplicial set**.

This notion generalizes the structure of the set $Sin_n(X)$. What we have done in the previous few pages is summarized as follows: to each topological space, we associate a semi-simplicial set $Sin_n(X)$. To each of these sets, we create the free abelian group $S_n(X)$ generated on these simplices. This forms a chain complex $(S_*(X), d)$, and we take the homology of said chain complex to form the homology of the space.

For a basic computation, consider the homology of the one point space $X = \{*\}$. There is a single n simplex for each n because X only has one point, which we will denote by C^n_*, so $S_*(X) = \mathbb{Z}\{C^n_*\}$. The composition of a face map with C^n_* is C^{n-1}_* because we are simply sending the simplex of one dimension smaller to $*$, so $d_i C^n_* = C^{n-1}_*$. But, this implies that:

$$dC^n_* = \begin{cases} C^{n-1}_* & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{15}$$

because for odd n we have an even number of face maps, so the alternating sum $d = \sum_{i=0}^{n} (-1)^n d_i$ cancels itself and vanishes. For even n we have one surviving d_i, so $dC^n_* = C^{n-1}_*$. So, the chain complex of n-chains on X is the following sequence (here I have labeled each copy of $\mathbb{Z}$ with which copy of $S_n(X)$ it is):

$$\begin{array}{ccccccccccc} \ldots \xrightarrow{id} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{0} & 0 & \longrightarrow \ldots \\ & S_3(X) & & S_2(X) & & S_1(X) & & S_0(X) & & & \end{array} \tag{16}$$

For the even and nonzero n, the kernel of $id : S_n(X) \cong \mathbb{Z} \to \mathbb{Z}$ is 0, so $Z_n(X) = 0$ and hence $H_n(X) = 0$. Similarly for odd n, the kernel of $0 : \mathbb{Z} \to \mathbb{Z}$ is $\mathbb{Z}$, but the image of the previous map is $im(id) = \mathbb{Z}$, hence $Z_n(X) = B_n(X) = \mathbb{Z}$, so the homology $H_n(X)$ vanishes as well. However, for $n = 0$, we note that $Z_0(X) = \mathbb{Z}$ and $B_0(X) = \{0\}$, hence $H_0(X) = \mathbb{Z}$. Hence to summarize:

$$H_n(*) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0 \end{cases} \tag{17}$$

We will now briefly consider the functorial nature of homology before moving on to lay out some definitions in category theory.

Definition 1.10 (Chain map)**.** Let C_* and D_* be two chain complexes. A **chain map** $f_* : C_* \to D_*$ is a collection of homomorphisms $f_n : C_n \to D_n$ such that the following diagram commutes:

$$\begin{tikzcd} ... \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d"] \arrow[d, "f_{n-1}"] & ... \\ ... \arrow[r, "d"] & D_{n+1} \arrow[r, "d"] & D_n \arrow[r, "d"] & D_{n-1} \arrow[r, "d"] & ... \end{tikzcd} \tag{18}$$

A chain map is just a sequence of maps which respect the differential. Now, we show that a continuous map from X to Y induces a chain map on the n-chains of X, which we will show in turn induces a morphism on homology.

Theorem 1.2. Let $f : X \to Y$ be a continuous map. Then f induces a chain map:

$$f_* : S_*(X) \to S_*(Y) \tag{19}$$

Proof. The induced map is very much a natural one, and we shall map the generators σ of $S_n(X)$ to $\sigma \mapsto f \circ \sigma : \Delta^n \to Y$, which extends to a homomorphism $f_* : S_*(X) \to S_*(Y)$ by linearity. We must show that f_* is in fact a chain map. We have:

$$d_i(f_*\sigma) = d_i(f \circ \sigma) = (f \circ \sigma) \circ d^i = f(d_i\sigma) \tag{20}$$

hence summing this up, we see that $df_* = f_*d$. $\square$

Theorem 1.3. Any chain map $f_* : C_* \to D_*$ induces a homomorphism on homology $f_* : H_*(C) \to H_*(D)$.

Proof. We will restrict each $f_n : C_n \to D_n$ to the n-cycles $Z_n(C)$ and show that the map $f_n : Z_n(C) \to Z_n(D)$ and is well defined. Suppose that $x \in Z_n(C)$, so $dx = 0$. Then $df_n(x) = f_{n-1}(dx) = 0$ because f_* is a chain map, hence $f_n(x) \in ker(d : D_n \to D_{n-1})$, and the map is well defined on the n-cycles. Projecting $Z_n(D)$ onto the homology with the map $\pi_D : Z_n(D) \to H_n(D)$, we have the diagram:

$$\begin{tikzcd} Z_n(C) \arrow[r, "f_n"] \arrow[d, "\pi_C"] & Z_n(D) \arrow[d, "\pi_D"] \\ H_n(C) \arrow[r, dashed, "\tilde{f}_n"] & H_n(D) \end{tikzcd} \tag{21}$$

and we will show that $\tilde{f}_n : H_n(C) \to H_n(D), [c] \mapsto \pi_D f_n(c)$, i.e. that we have a well defined lift from the quotient. Suppose that $[c] = [c'] \in H_n(C)$, so $c = c' + db$ for $b \in C_{n+1}$. Then we must show that $\tilde{f}([c]) = \tilde{f}([c']) + \tilde{f}([db]) = \tilde{f}([c'])$, so we have:

$$\tilde{f}([db]) = \pi_D f_n(db) = \pi_D(df_{n+1}b) = 0 \tag{22}$$

because $df_{n+1}b \in B_n(D) = im(d : D_{n+1} \to D_n)$. Hence the map is well defined, so we have a valid morphism $H_n(C) \to H_n(D)$ for all n. $\square$

2. Categorical Considerations

So, we have seen that any continuous map $f : X \to Y$ induces a well defined map $f_* : H_*(X) \to H_*(Y)$ on the homology. In other words, $H_n(\cdot)$ **is a covariant functor from** Top **to** Ab. We will be using many of its functorial properties, and now may be a good time to brush up on your category theory. We will recall a few definitions which will be important to us. We denote the objects of a category $\mathcal{C}$ by $obj(\mathcal{C})$, and the morphisms from X to Y by $\mathcal{C}(X,Y)$.

Definition 2.1 (Natural transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\Theta : F \to G$ consists of, for each $X \in obj(\mathcal{C})$, a map $\Theta_X : F(X) \to G(X)$ such that for each $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes (in the category $\mathcal{D}$).

$$\begin{tikzcd} F(X) \arrow[r, "\Theta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Theta_Y"] & G(Y) \end{tikzcd} \tag{23}$$

A natural transformation is essentially a way of mapping one functor into another while respecting the structure of the functor; it is essentially a "morphism" of functors. A functor takes you between categories $\mathcal{C} \to \mathcal{D}$, but a natural transformation takes you between functors. The canonical example of this is that for a small category $\mathcal{C}$ (recall $\mathcal{C}$ is small if $obj(\mathcal{C})$ is a set), we can define the category of functors $Fun(\mathcal{C}, \mathcal{D})$. The objects of this category are functors $F : \mathcal{C} \to \mathcal{D}$, and the morphisms of this category are natural transformations $\Theta : F \to G$. We also recall the definition of the opposite category:

Definition 2.2 (Opposite category)**.** For a category $\mathcal{C}$, define its **opposite category** $\mathcal{C}^{op}$ to be the category with:

- Objects $obj(\mathcal{C}^{op}) = obj(\mathcal{C})$.
- Morphisms $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$.

We can formalize some of the notions that we've discussed about simplices by defining the **simplex category** Δ. This category has the following data:

- **Objects** are the sets $[n] := \{0, 1, ..., n\}$.
- **Morphisms** are weakly order preserving maps $\phi : [n] \to [m]$. By weakly order preserving, we mean that $i < j \implies \phi(i) \leq \phi(j)$.

With this notation, the face maps are maps $d^i : [n-1] \to [n]$

$$d^i(k) = \begin{cases} k & k < i \\ k+1 & k \geq i \end{cases} \tag{24}$$

The counterpart of the face maps are **degeneracy maps** $s^i : [n+1] \to [n]$ which repeat the value i are:

$$s^i(k) = \begin{cases} k & k \leq i \\ k-1 & k > i \end{cases} \tag{25}$$

Now, we can define simplicial objects.

Definition 2.3 (Simplicial object)**.** Let $\mathcal{C}$ be a category. A **simplicial object** in $\mathcal{C}$ is a functor $K : \Delta^{op} \to \mathcal{C}$. A **semi-simplicial object** is a functor $K : \Delta^{op}_{inj} \to \mathcal{C}$, where Δ_{inj} denotes the subcategory of Δ whose morphisms are injective. We use the phrase **simplicial set** to mean a simplicial object in Set.

Suppose that we fix a space X. Then the set of simplicial complexes in X is a functor:

$$Sin_* : \Delta^{op} \to Set \tag{26}$$

On objects of these categories, this assigns to each simplex $[n] = \Delta^n$ the set $Sin_n(X) = Top(\Delta^n, X)$. The reason this is from Δ^{op} to Set is because of how morphisms are mapped by this functor. For the face map $d^i : \Delta^{n-1} = [n-1] \to \Delta^n = [n]$, we induce a morphism:

$$d_i = Sin_*(d^i) : Sin_n(X) = Sin_*([n]) \to Sin_{n-1}(X) = Sin_*([n-1]) \tag{27}$$

i.e. Sin_* reverses the arrows between objects, as seen in the following diagram:

$$\begin{tikzcd} {[n-1]} \arrow[r, "d^i"] \arrow[d, "Sin_*"] & {[n]} \arrow[d, "Sin_*"] \\ Sin_{n-1}(X) & Sin_n(X) \arrow[l, "d_i"] \end{tikzcd} \tag{28}$$

We denote the category of simplicial sets to be **sSet**. This category has the following data:

- Objects are simplicial sets, i.e. functors $\Delta^{op} \to Set$.
- Morphisms are the natural transformations between these functors.

Now, we consider certain types of maps. We will use this shortly to define reduced homology.

Definition 2.4 (Split epimorphism)**.** A morphism $f : X \to Y$ in $\mathcal{C}$ is called a **split epimorphism** if there is $g : Y \to X$ (such a g is called a **section** of f) such that:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "1_Y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \tag{29}$$

A morphism $g : Y \to X$ in $\mathcal{C}$ is called a **split monomorphism** if there is $f : Y \to X$ which makes the above diagram commute.

Lemma 2.1. A morphism is an isomorphism iff it is a split epimorphism and a split monomorphism.

Lemma 2.2. Any functor sends split epimorphisms to split epimorphisms and split monomorphisms to split monomorphisms.

Epimorphisms are the categorical generalization of surjective morphisms, and monomorphisms generalize injectivity. For example, if $\mathcal{C} = Set$, then an epimorphism is exactly a surjective map. Let us examine the category $\mathcal{C} = Ab$ of abelian groups. Suppose that $f : A \to B$ is a split epimorphism. Then we have a section $g : B \to A$, and we can consider the inclusion $\iota : ker(f) \hookleftarrow A$. Then we have an isomorphism:

$$\iota \oplus g : ker(f) \oplus B \xrightarrow{\sim} A \tag{30}$$

This should be reasonably intuitive because a split epimorphism is a surjection, so $B \cong A/ker(f)$, and the presence of a section means we can actually solve the extension problem in this case.

We now put these ideas into practice. First recall the definition of terminal and initial objects in a category.

Definition 2.5. Let $\mathcal{C}$ be a category. A **terminal object** is an object $X \in obj(\mathcal{C})$ such that for each $A \in obj(\mathcal{C})$, there is a morphism (which is unique) $A \to X$. An **initial object** is an object X such that for each $A \in obj(\mathcal{C})$, there is a morphism $X \to A$.

In the category **Top**, an initial object is $\emptyset$ and a terminal object is the one-point space $\{*\}$ (which we often denote by $*$). So, let X be a topological space. Then we get a map:

$$\xi_n : H_n(X) \longrightarrow H_n(*) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases} \tag{31}$$

because H_n is a covariant functor. On the 0-chains in X, note that $S_0(X)$ is generated by the points in X and each element $c \in S_0(X)$ is of the form $c = \sum_{i=1}^m a_i x_i$ with $a_i \in \mathbb{Z}$ and $x_i \in X$. So, we get a map $S_0(X) \to S_0(*), c \mapsto \sum_{i=1}^m a_i$, which induces the map $\xi_0 : H_0(X) \to H_0(*)$.

If $X \neq \emptyset$, then $X \to *$ is a split epimorphism, so ξ_n is a split epimorphism as well. Thus we must be able to factor its kernel out as a summand:

$$H_n(X) \cong H_n(*) \oplus ker(\xi_n) \tag{32}$$

We will call this kernel the reduced homology group of X.

Definition 2.6 (Reduced homology group)**.** Let $X \neq \emptyset$ be a topological space. Then the **reduced nth homology group** of X is defined as $\tilde{H}_n(X) = ker(\xi_n)$, and the homology splits as a sum:

$$H_*(X) = H_*(*) \oplus \tilde{H}_*(X) \tag{33}$$

In particular, note that the homology is the reduced homology for each $H_n(X)$ as long as $n > 0$. For the $n = 0$ case, the reduced homology factors out a copy of $\mathbb{Z}$:

$$H_0(X) = \mathbb{Z} \oplus \tilde{H}_0(X) \tag{34}$$

The reduced homology essentially compensates for the fact that $*$ has a nontrivial 0th homology group, so we can extract only the interesting mathematics from more complicated spaces and not be required to carry a copy of $\mathbb{Z}$ with us everywhere we go.

Note that the interpretation of the 0th homology group is that it characterizes the path-components of a space. In this case because $*$ has a single path component, it has $H_0(*) = \mathbb{Z}$. On the other hand, a space like $S^1 \coprod S^1$ has $H_0(S^1 \coprod S^1) = \mathbb{Z} \oplus \mathbb{Z}$ because it has two path components.

We now consider the effect of homotopy on the homology groups of the space. Like the fundamental group, we will show that the homotopy groups $H_n(X)$ are invariant under homotopy, which offers another way to categorize spaces, just like $\pi_1(X)$.

3. Homotopy

Recall the definition of a homotopy. If you are unfamiliar with this concept, it may be useful to check out my notes on the subject.

Definition 3.1 (Homotopy)**.** Let $f, g : X \to Y$ be continuous maps. A **homotopy** from f to g is a continuous map:

$$h : X \times I \to Y \tag{35}$$

where $I = [0,1]$ is the unit interval, such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for each $x \in X$. We say that f and g are **homotopic**, and h is a **homotopy** between them. We denote this by $f \simeq_h g$.

Note that $\simeq$ is an equivalence relation on functions. We let $[X,Y]$ denote the homotopy classes of maps from X to Y:

$$[X,Y] := Top(X,Y)/\simeq \tag{36}$$

As in studying the fundamental group, homotopy will be an essential tool for our study of homology. Namely (although we will not prove this until later), homology is invariant under homotopy, and so if $H_*(X) \neq H_*(Y)$, then the spaces X and Y cannot be homotopy equivalent, much less so homeomorphic.

Theorem 3.1 (Homotopy invariance of homology)**.** View $H_*(\cdot)$ as a functor from **Top** to **Ab**. If $f_0 \simeq f_1 : X \to Y$, then:

$$H_*(f_0) = H_*(f_1) : H_*(X) \to H_*(Y) \tag{37}$$

and homology cannot distinguish between homotopic maps.

Definition 3.2. The **homotopy category** of topological spaces **Ho(Top)** is the category with:

- Objects: Same as **Top**.
- Morphisms: $\mathbf{Ho(Top)}(X,Y) := [X,Y]$

Definition 3.3 (Homotopy equivalence)**.** A map $f : X \to Y$ is a **homotopy equivalence** if $[f] \in [X,Y]$ is an isomorphism. Equivalently, f is a homotopy equivalence if there is $g : Y \to X$ such that $[f \circ g] = [1_Y]$ and $[g \circ f] = [1_X]$. In this case, we call g a **homotopy inverse** of f, and we say that X and Y are **homotopy equivalent**.

Note that because of the homotopy invariance of homology, $H_* : \mathbf{Top} \to \mathbf{Ab}$ factors through **Ho(Top)** as $\mathbf{Top} \to \mathbf{Ho(Top)} \to \mathbf{Ab}$. An easy corollary of the homotopy invariance of homology is that homotopy equivalent spaces have the same homology groups, as seen below.

Corollary 3.1.1. Homotopy equivalence induces an isomorphism in H_*, i.e. if X is homotopy equivalent to Y, then they have the same homology groups.

Proof. If $f : X \to Y$ and $g : Y \to X$ are homotopy inverses, then $H_*(f) \circ H_*(g) = H_*(id_X) = 1_{H_*(X)}$, and similarly for the other way. So, $H_*(f)$ and $H_*(g)$ are inverses of one another and hence isomorphisms. □

A nice diagram to remember when dealing with the homotopy invariance of homology is the following commutative square of compositions (where π is the quotient map taking $\mathbf{Top}(X,Y) \to [X,Y]$):

$$\begin{CD} \mathbf{Top}(X,Y) \times \mathbf{Top}(Y,Z) @>{\circ}>> \mathbf{Top}(X,Z) \\ @V{\pi}VV @VV{\pi}V \\ [X,Y] \times [Y,Z] @>{\circ}>> [X,Z] \end{CD} \tag{38}$$

Homotopy equivalence is a nicer type of map in many ways than homormorphism, simply because it is weaker. However, note that it does not preserve many nice topological properties of spaces, including compactness, metrizability, being Hausdorff, second countability, and dimension (as a topological manifold). The essential reasoning behind this is that homotopy equivalence allows one to crush a space into a point, as long as there are no holes in the region of interest. It only preserves the structure of the space in the broadest sense, and if there are parts of the space that may deformation retract onto other parts (like a dangling fiber), then no information about this need be preserved under homotopy equivalence.

To directly illustrate an example of one of these properties being lost, consider the sphere $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$. Then $\mathbb{R}^n \setminus \{0\}$ deformation retracts onto S^{n-1} because the projection map $v \to \frac{v}{||v||}$ onto the sphere is a homotopy equivalence. However, the space $\mathbb{R}^n \setminus \{0\}$ is certainly not a compact space, but S^{n-1} is a compact space. Hence when working with homotopy equivalence, it is essential to remember that it is not nearly as strong a map as a homeomorphism, and that some properties may be lost in translation from one space to another.

Definition 3.4. A space X is **contractible** if the unique map $X \to \{*\}$ is a homotopy equivalence.

Corollary 3.1.2. Let X be contractible. Then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) \cong 0$ for $n > 0$, i.e. we have:

$$\tilde{H}_*(X) = 0 \tag{39}$$

If a space is contractible, it has "no holes in its surface", and can be crushed down into the one point space by homotopy. This implies that it has both a trival fundamental group and trivial reduced homology groups. However, note that the converse is not true: if a space has trivial reduced homology groups, it need not be contractible. This is closely related to the Poincaré conjecture, and because the proof of this is extremely sophisticated we will not have the tools to prove this in these notes. Another way to formulate contractibility is if X can deformation retract onto a point, which we will define now.

Definition 3.5 (Deformation retract)**.** Let X be a space. An inclusion $\iota : A \hookrightarrow X$ is a **deformation retract** if there the homotopy relative to A from the identity into a map $X \to A$, i.e. we have a map $h : X \times I \to A$ such that the following hold:

1. $h(x, 0) = x$ for each $x \in X$.
2. $h(a, t) = a$ for $a \in A$ and $t \in I$.
3. $h(x, 1) \in A$ for each $x \in X$.

We will now work towards a proof of the homotopy invariance of homology. To do this, we take a brief detour into the concept of a chain homotopy.

Definition 3.6 (Chain homotopy)**.** Let C_*, D_* be chain complexes and $f_*, g_* : C_* \to D_*$ be chain maps. Then a **chain homotopy** $h_* : f_* \simeq g_*$ is a chain map $h_n : C_n \to D_{n+1}$:

$$\begin{tikzcd}
... \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "g_{n+1}"', "f_{n+1}"] & C_n \arrow[r, "d"] \arrow[d, "g_n"', "f_n"] \arrow[dl, "h_n"] & C_{n-1} \arrow[r, "d"] \arrow[d, "g_{n-1}"', "f_{n-1}"] \arrow[dl, "h_{n-1}"] & ... \\
... \arrow[r, "d"] & D_{n+1} \arrow[r, "d"] & D_n \arrow[r, "d"] & D_{n-1} \arrow[r, "d"] & ...
\end{tikzcd} \tag{40}$$

such that:

$$hd + dh = g - f \tag{41}$$

Chain homotopy is an equivalence relation on the space of maps between topological spaces, which is easy to verify straight from the definition. The reason we are considering chain homotopy is because chain maps which are chain homotopic will induce the same maps on homology.

Theorem 3.2. If $f_0 \simeq f_1 : C_* \to D_*$ are chain homotopic, then they induce the same maps on homology:

$$(f_0)_* = (f_1)_* : H_*(C_*) \to H_*(D_*) \tag{42}$$

Proof. Recall that the induced map is the natural one, $f_*([x]) = [f(x)]$ for a map $f : C_* \to D_*$. So, we must show that for $x \in Z_n(C_*)$, $f_0(x)$ and $f_1(x)$ are in the same coset when we mod $Z_n(D_*)$ out by $B_n(D_*)$ to form the homology, i.e. $f_0(x) - f_1(x) = da$ with $a \in D_{n+1}$. Let $x \in Z_n(C_*)$. Then:

$$f_0(x) - f_1(x) = (dh + hd)(x) = d(hx) + h(dx) = d(hx) \in B_n(D_*) \tag{43}$$

because $dx = 0$ for a cycle x. Thus these induce the same maps on homology. □

3.1. **Proof of the Homotopy Invariance of Homology.** Using the above theorem, we can reduce proving the homotopy invariance of homology to showing that homotopic maps $f, g : X \to Y$ induce chain homotopic maps $f_*, g_* : S_*(X) \to S_*(Y)$ on the n-chains, since these will by default induce the same maps on homology. We will need the following lemma:

Lemma 3.3. Let $k : f \simeq g$ be a chain homotopy between $f, g : C_* \to D_*$ and let $j : D_* \to E_*$ be a chain map. Then $j \circ k : j \circ f \simeq j \circ g$ is a chain homotopy between $j \circ f$ and $j \circ g$.

Proof. This is immediate once you draw out the diagrams. Because j is a chain map, $dj = jd$. So, $(jh)d + d(jh) = jhd + jdh = j(hd + dh) = j(f - g) = jf - jg$ because $hd + dh = f - g$ as h is a chain homotopy. □

Now, consider an arbitrary homotopy $h : X \times I \to Y$ from $f : X \to Y$ to $g : X \to Y$. Note that we have canonical injections $\iota_0 : X \times \{0\} \hookrightarrow X \times I$ and $\iota_1 : X \times \{1\} \hookrightarrow X \times I$ such that the diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \arrow[dr, "\iota_0"] \arrow[drr, bend left, "f"] & & \\
& X \times I \arrow[r, "h"] & Y \\
X \times \{1\} \arrow[ur, "\iota_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd} \tag{44}$$

We are trying to prove that we can find a chain homotopy $h_* : f_* \to g_* : H_*(X) \to H_*(Y)$ to prove the homotopy invariance. Since H_* is a covariant functor, note that $f_* = h_* \circ \iota_0$ and $g_* = h_* \circ \iota_1$. Using the lemma, if we show that we can find a chain homotopy $k_* : (\iota_0)_* \simeq (\iota_1)_*$, then we can compose the chain homotopy with h_* to get a chain homotopy $h_* \circ k_* : f_* \simeq g_*$. Thus, if we can find a map:

$$k_n : S_n(X) \to S_{n+1}(X \times I) \qquad\qquad dk + kd = (\iota_0)_* - (\iota_1)_* \tag{45}$$

then we have proved the theorem. To construct such a map, view $X \times I \cong X \times \Delta^1$. We can view k as a map $k : \mathbf{Top}(\Delta^n, X) \to \mathbf{Top}(\Delta^{n+1}, X \times \Delta^1)$, and if we can find a way to construct a map:

$$\gamma : \Delta^{n+1} \to \Delta^n \times \Delta^1 \tag{46}$$

then given $\sigma \in \mathbf{Top}(\Delta^n, X)$, we can construct a map $\tau \in \mathbf{Top}(\Delta^{n+1}, X)$, such that:

$$\begin{tikzcd}
\Delta^{n+1} \arrow[r, "\gamma"] \arrow[dr, bend right, "\tau"'] & \Delta^n \times \Delta^1 \arrow[d, "\sigma"] \arrow[d, hook] \\
& X \times I
\end{tikzcd} \tag{47}$$

To do this, we will take for granted a theorem that will be proven later in these notes in Section 4: the existence of a **cross product** on homology.

Theorem 3.4. There is a natural bilinear map $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, called the **cross product**, such that the following conditions hold.

(1) $\times$ is natural, so given maps $f : X \to X'$ and $g : Y \to Y'$, the following diagram commutes:

(48)
$$\begin{array}{ccc} S_p(X) \times S_q(Y) & \xrightarrow{\times} & S_{p+q}(X \times Y) \\ \downarrow{\scriptstyle (f_*, g_*)} & & \downarrow{\scriptstyle (f \times g)_*} \\ S_p(X') \times S_q(Y') & \xrightarrow{\times} & S_{p+q}(X' \times Y') \end{array}$$

where f_* and g_* are the maps induced on the n-chains and $(f \times g)_*$ is the map induced by $f \times g : X \times Y \to X' \times Y', (x, y) \mapsto (f(x), g(y))$.

(2) $\times$ is bilinear, i.e. $(a + a') \times b = a \times b + a' \times b$ and $a \times (b + b') = a \times b + a \times b'$.

(3) d is an *anti-derivation* with respect to $\times$, so

(49)
$$d(a \times b) = da \times b + (-1)^p a \times db$$

(4) $\times$ is normalized. For $x \in X$, $y \in Y$, and each $a \in S_p(X), b \in S_q(Y)$, given the inclusions $j_x : Y \hookrightarrow X \times Y, y \mapsto (x, y)$ and $\iota_y : X \to X \times Y$, the induced maps $(j_x)_* : S_*(Y) \to S_*(X \times Y)$ and $(\iota_y)_* : S_*(X) \to S_*(X \times Y)$ satisfy the following equations:

(50)
$$(j_x)_*(b) = C_x^0 \times b$$

(51)
$$(\iota_y)_*(a) = a \times C_y^0$$

Using the cross product, we can explicitly construct a chain homotopy $k_* : (\iota_0)_* \simeq (\iota_1)_*$. Pick a map $j : \Delta^1 \to I$ such that $d_0 j = C_1^0$ and $d_1 j = C_0^0$, which is a very obtuse way of saying $j(0) = 1$ and $j(1) = 0$, i.e. j is an orientation reversing path[1]. Then define our map $k_n : S_n(X) \to S_{n+1}(X \times I)$ as follows:

(52)
$$k_n \sigma := (-1)^n \sigma \times j$$

Now we take the boundary of this map. Using the properties of $\times$, we have:

$$\begin{aligned} d(k_n \sigma) = (-1)^n d(\sigma \times j) = (-1)^n (d\sigma \times j + \sigma \times dj) &= -(-1)^{n-1}(d\sigma \times j) + \sigma \times (d_0 j - d_1 j) \\ &= -k_{n-1}(d\sigma) + \sigma \times C_1^0 - \sigma \times C_0^0 \end{aligned}$$

where we use that $k_{n-1}(d\sigma) = (-1)^{n-1} d\sigma \times j$. Now, using the normalization of the cross product for the induced maps $(\iota_0)_*, (\iota_1)_* : S_n(X) \to S_n(X \times I)$, we can write[2] $\sigma \times C_1^0 = (\iota_1)_*(\sigma)$ and $\sigma \times C_0^1 = (\iota_0)_*(\sigma)$. Thus we see that:

(53)
$$dk\sigma = -kd\sigma + ((\iota_1)_* - (\iota_0)_*)\sigma$$

and we see that k_* is indeed a chain homotopy from $(\iota_0)_*$ to $(\iota_1)_*$, which completes the proof.

Note that there is still a black box in this proof: namely, proof of the existence of the cross product. We will prove this in a later chapter when we discuss it further in detail. The proof ends up requiring the fact that the product $\Delta^p \times \Delta^q$ has a trivial $p + q - 1$ homology group; this can either be shown explicitly, or one can prove a weaker version of the homotopy invariance of homology for star shaped regions, since $\Delta^p \times \Delta^q$ is convex and hence star shaped. However, this is not difficult to prove and can be shown by considering the straight line homotopy.

---

[1] Recall that C_p^n is the constant map $\Delta^n \to X$ with image $p \in X$.

[2] These are exactly the maps $\iota_y : X \to X \times I$ in the theorem at $y = 0$ and $y = 1$.

4. Sequences and Relative Homology

The tone of these notes is now going to pivot quite a bit and begin preparation for homological algebra, which was developed originally to aid in the study of homology and cohomology. We will be considering abstract sequences of objects and drawing lots of arrows, and our proofs will take a more diagrammatic tone than previously. We begin with the definition of a sequence.

Definition 4.1 (Sequence)**.** A **sequence** of abelian groups is a diagram of the form:

$$... \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow ... \tag{54}$$

such that $f_n \circ f_{n-1} = 0$ for each n. A sequence is **long exact** if $im(f_n) = ker(f_{n-1})$ for each n, i.e. if its homology vanishes identically.

In other words, a sequence is a chain complex, and for any such chain complex $im(f_n) \subseteq ker(f_{n-1})$, the central property that enables the definition and study of homology. If the sequence only has a finite number of nonzero elements, it is understood that we have just left out the 0 elements after the last nonzero one.

Definition 4.2. A **short exact sequence** (SES) is a long exact sequence of the form:

$$0 \longrightarrow A \overset{\iota}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C \longrightarrow 0 \tag{55}$$

Note here we have drawn the arrows in a certain way to suggest specific properties. Namely, for the first and last arrows to be exact, then A must embed into B and B must project onto C. Additionally, if this sequence is exact then $A \cong ker(p)$, because ι embeds A as $A \cong im(\iota) \subseteq B$ and $im(\iota) = ker(p)$. Similarly, we must have $C \cong coker(\iota)$, because $coker(\iota) = B/im(\iota) \cong B/ker(p) \cong C$ as p is surjective. Recall the cokernel of a map is defined as:

$$coker(f : A \to B) := B/im(f) \tag{56}$$

Intuition for the cokernel of a map is a bit more subtle than intuition for the kernel. This is because the cokernel is not strictly a group (or module, etc.) as we do not necessarily have $im(f) \trianglelefteq B$, and so we cannot consider its algebraic properties. However, an easy way to view the cokernel is by considering how much f deviates from being a surjection. Just as how if the kernel of f grows larger, it is less close to being injective, we have a similar property for the cokernel: as the cokernel grows, f is less and less close to being a surjection. In particular, f is a surjection iff $coker(f) = 0$, so this gives a nice analog to how a map is injective iff $ker(f) = 0$.

Definition 4.3. We define $\mathbf{Top_2}$ to be the category of pairs (X, A) where X is a topological space and A is a subspace. The morphisms $(X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subseteq B$.

Note that **Top** is a subcategory in $\mathbf{Top_2}$, which can be seen by either sending $X \mapsto (X, \emptyset)$ or $X \mapsto (X, X)$.

Lemma 4.1. Let B_* be a chain complex with subchain complex $A_* \subseteq B_*$. Then there is a unique structure of a chain complex on the graded abelian groups $C_* := B_*/A_*$ such that the canonical projection $B_* \to C_*$ is a chain map.

Proof. The following diagram outlines the situation we are in, where the diagram is exact vertically and horizonatally:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d"] \arrow[d, hook] & A_n \arrow[r, "d"] \arrow[d, hook] & A_{n-1} \arrow[r, "d"] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d"] \arrow[d, two heads, "\pi"] & B_n \arrow[r, "d"] \arrow[d, two heads, "\pi"] & B_{n-1} \arrow[r, "d"] \arrow[d, two heads, "\pi"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] & C_{n-1} \arrow[r, "d"] & \cdots
\end{tikzcd} \tag{57}$$

We need to define the maps $d : C_n \to C_{n-1}$ to make (C_*, d) into a chain complex, where each $C_n = B_n/A_n$. Define:

$$d([b]) := [db] \tag{58}$$

To show this is well defined, suppose that $b = b' + a$ with $b, b' \in B_n$ and $a \in A_n$ (i.e. $[b] = [b']$). Then:

$$db = db' + da \implies [db] = [db'] \tag{59}$$

because $[da] = 0$ as d is a chain map on the complex A_*, so $da \in A_{n-1}$ which is killed by the quotient. This is indeed a chain complex, because:

$$d^2[b] = d[db] = [d^2 b] = 0 \tag{60}$$

Finally, this definition makes $\pi : B_* \to C_*$ into a chain map by definition, because $\pi(b) = [b]$. Furthermore, this makes it clear the definition is unique, so we are done. □

Definition 4.4 (Relative singular chain)**.** A **relative singular chain** is a functor $S_* : \mathbf{Top_2} \to \mathbf{Chain}$, where **Chain** is the category of chain complexes, which maps:

$$S_*(X, A) := S_*(X)/S_*(A) \tag{61}$$

which is well defined as $S_*(A)$ is a subcomplex of $S_*(X)$ for a subspace A of X.

Definition 4.5 (Relative Homology)**.** The **relative homology** of $(X, A) \in \mathbf{Top_2}$ is:

$$H_*(X, A) := H_*(S_*(X, A)) \tag{62}$$

Note that the relative homology **is not** the quotient of the homology groups. However, it is related, and we will show that in certain situations the relative homology behaves just like a quotient.

As an example, consider $X = \Delta^n$ and $A = \partial X = \partial\Delta^n$. Then the identity $\iota : \Delta^n \to \Delta^n$ is an element of $Sin_n(\Delta^n$, and the boundary satisfies $d\iota \in S_*(\partial\Delta^n) = S_*(A)$. So, $[\iota]$ is a relative cycle ($[\iota] \in Z_n(X, A)$) because $d[\iota] = [d\iota] = 0$ in $S_*(X, A) = S_*(X)/S_*(A)$, hence $[\iota]$ defines an element of the relative homology $H_n(\Delta^n, \partial\Delta^n)$, which is interesting because ι is not a representative of an element in the homology of Δ^n. Eventually, we will see that $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ with $[\iota]$ as a generator.

Suppose we have a short exact sequence of chain complexes

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0 \tag{63}$$

In particular, this is the case when $A_* = S_*(A)$, $B_* = S_*(X)$, and $C_* = S_*(X, A)$ for $(X, A) \in \mathbf{Top_2}$, so anything proved with a sequence of this form can be used for relative simplicial homology as well. When we apply the functor $H_*(\cdot)$, we unfortunately do not get a short exact sequence, although some parts of this SES stay exact when dropped to the level of homology. We make this precise in the following theorem, which will be very important as we continue our study of homology.

Theorem 4.2. Let $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ be a short exact sequence of chain complexes. Then there is a natural homomorphism:

$$\partial : H_n(C_*) \to H_{n-1}(A_*) \tag{64}$$

such that the following sequence is long exact:

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(A_*) \arrow[r] & H_{n+1}(B_*) \arrow[r] & H_{n+1}(C_*) \arrow[dll, "\partial"] \\ & H_n(A_*) \arrow[r] & H_n(B_*) \arrow[r] & H_n(C_*) \arrow[dll, "\partial"] \\ & H_{n-1}(A_*) \arrow[r] & H_{n-1}(B_*) \arrow[r] & H_{n-1}(C_*) \arrow[r] & \cdots \end{tikzcd} \tag{65}$$

Proof. We begin by showing exactness at the $\to H_n(B_*) \to$ joint. Let $\bar{f}_*, \bar{g}_*$ be the maps induced on the homology by f_*, g_*, i.e. $\bar{f}([a]) := [f(a)]$. Then:

$$\bar{g}_n \circ \bar{f}_n([a]) = [g \circ f(a)] = [0] = 0 \tag{66}$$

so $im(\bar{f}_*) \subseteq ker(\bar{g}_*)$. The reverse inclusion is a bit more involved, and we will need to chase the following diagram, which is the statement that the above SES is exact (in the future I will not draw this diagram explicitly).

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ \cdots \arrow[r, "d"] & A_{n+1} \arrow[r, "d"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "d"] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r, "d"] & B_{n+1} \arrow[r, "d"] \arrow[d, "g_{n+1}"] & B_n \arrow[r, "d"] \arrow[d, "g_n"] & B_{n-1} \arrow[r, "d"] \arrow[d, "g_{n-1}"] & \cdots \\ \cdots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d] & C_n \arrow[r, "d"] \arrow[d] & C_{n-1} \arrow[r, "d"] \arrow[d] & \cdots \\ & 0 & 0 & 0 & \end{tikzcd} \tag{67}$$

Pick $[b] \in ker(\bar{g}_n)$. Then $[g_n b] = 0$ in $H_n(C)$, so $g_n b = dc$ for $c \in C_{n+1}$. Lift c to $b' \in B_{n+1}$, so $c = g_{n+1}b'$. Then $g_n b = dc = dg_{n+1}b' = g_n db'$, so $g_n(b - db') = 0$, which implies $b - db' = f_n a$ for $a \in A_n$. Then it is clear that $[b] = [db'] + [f_n a] = \bar{f}_n([a])$, so $im(\bar{f}_*) = ker(\bar{g}_*)$.

We now turn to the connecting homomorphism ∂, which we define via a diagram chase. Let $[c] \in H_n(C_*)$, so $c \in C_n$ and $dc = 0$. Lift c to $b \in B_n$, so $g_{n-1}db = dg_n b = dc = 0 \implies db \in ker(g_{n-1}) = im(f_{n-1})$, so we can lift db to $a \in A_{n-1}$ so $f_{n-1}a = db$. We use this element a to define $\partial[c]$ as $\partial[c] := [a]$.

Now, we show this is well defined. Suppose $[c] = [c']$, so $c = c' + dc''$ with $c'' \in C_{n+1}$. Let $b' \in B_n$ be a lift of c' and $b'' \in B_{n+1}$ be a lift of c'', so $c' = g_n b'$, $c'' = g_{n+1}b''$, and $dc'' = g_n(b' - b)$. Applying dg_{n+1} to b'', we see $dg_{n+1}b'' = dc'' = g_n(b' - b)$, but because g_* is a chain map, we also have $g_n db'' = dg_{n+1}b''$, hence $g_n(b' - b - db'') = 0$ and we lift $b' - b - db''$ to $a'' \in A_n$. Note that $db = f_{n-1}(a)$ and $db' = f_{n-1}(a')$, where a' is defined for b' as above. But then $df_n(a'') = d(b' - b - db'') = db' - db$, and also $df_n(a'') = f_{n-1}da''$, and by the definition of a and a' we have $db' - db = f_{n-1}(a - a') = f_{n-1}(da'') \implies f_{n-1}(a - a' - da'') = 0$. But f_{n-1} is injective, so this implies $a = a' + da''$, so $[a] = [a']$ and ∂ is thus well defined.

Finally, we will prove exactness at the $H_n(C_*)$ vertex. Suppose $[c] \in H_n(C_*)$ and $[c] = \bar{g}_n([b])$, so $[c] = [g_n(b)]$ for $b \in B_n$ with $db = 0$. Then when we lift c to B_n, we can lift it to b. But, $db = 0$, so when we lift db in our definition of δ, we lift it to $a \in ker(f_{n-1})$. But $ker(f_{n-1}) = 0$ because it is injective, hence $a = 0$, and thus $\partial[c] = 0$, which shows $im(\bar{g}_*) \subseteq ker(\partial)$.

Suppose $[c] \in H_n(C_*)$ and $\partial[c] = 0$. We must show we can lift $[c]$ to $[b]$. Note that although we can lift c to $b \in B_n$ (as we did in the definition of ∂) we do not necessarily have that $db = 0$, so we must find a lift $b' \in B_n$ of c with $db' = 0$. Let $\partial[c] = [\tilde{a}]$ with $\tilde{a} \in A_{n-1}$. Then c lifts to $b \in B_n$ by the definition of ∂, and we have $db = f_{n-1}\tilde{a}$. Because $\partial[c] = [\tilde{a}] = 0$, we have $\tilde{a} = da$ with $a \in A_n$. So, define $b' := b - f_n a$. Then $db' = db - df_n a = db - f_{n-1}\tilde{a} = 0$, and $g_n b' = g_n b + g_n f_n a = g_n b = c$, hence we have found the desired lift. □

This "long exact sequence in homology" is the basis for many of the computational tools that we will study to calculate the homology of different spaces. In particular, this works very nicely with elements of $\mathbf{Top_2}$. Consider a pair $(X, A) \in \mathbf{Top_2}$. Then we have the canonical short exact sequence:

$$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0 \tag{68}$$

by definition of $S_*(X, A)$ as the quotient $S_*(X)/S_*(A)$, so we get a long exact sequence:

$$\cdots \longrightarrow H_{n+1}(A) \longrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X,A) \longrightarrow \cdots \tag{69}$$

which in many cases provides us a practical way to compute the reduced homology if we know the homology of one or both of the spaces. For example, consider the spaces $X = D^n$ and $A = S^{n-1}$. We know D^n is contractible, so $H_*(D^n) = *$, where $*$ is a chain complex concentrated in degree 0 (i.e. the homology of a point). Because of this, the middle vertices in the above diagram are 0, and so we get exact sequences:

$$0 \to H_q(D^n, S^{n-1} \to H_{q-1}(S^{n-1}) \to 0 \tag{70}$$

which implies that $H_q(D^n, S^{n-1}) \cong H_{q-1}(S^{n-1})$. This is a very nontrivial realization– note in particular that determining this had *nothing to do with the boundary map* ∂, and we only needed to use the fact that it existed and was well defined. This will be a recurring theme in many areas of algebraic topology– many times we only care about the exactness of maps, not about the maps themself. Because of this, many theorems that we studied in homological algebra will be applicable to our study of algebraic topology. In particular, recall the five lemma.

Lemma 4.3 (Five Lemma)**.** Suppose the following diagram commutes and the rows are exact.

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array} \tag{71}$$

(1) If f_5 is injective and f_2, f_4 are surjective, then f_3 is surjective.
(2) If f_1 is surjective and f_2, f_4 are injective, then f_3 is injective.

In particular, if f_1 is surjective, f_5 is injective, and f_2, f_4 are isomorphisms, then f_3 is an isomorphism.

Although there seem to be a lot of conditions on this lemma, we will typically only use it when f_3 is sandwiched by isomorphisms– this condition will appear a lot in proofs later on in the course. The following immediate consequence of this lemma will be useful.

Corollary 4.3.1. Suppose we have the following diagram of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r] \arrow[d, "f_1"] & B_* \arrow[r] \arrow[d, "f_2"] & C_* \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & C'_* \arrow[r] & 0
\end{tikzcd}$$

If any two of $f_1, f_2, or f_3$ are isomorphisms on homology, then so is the third.

Proof. By drawing out the induced long exact sequence and maps on homology, we see:
(72)

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(B_*) \arrow[r] \arrow[d, "f_{2*}"] & H_{n+1}(C_*) \arrow[r, "\partial"] \arrow[d, "f_{3*}"] & H_n(A_*) \arrow[r] \arrow[d, "f_{1*}"] & H_n(B_*) \arrow[r] \arrow[d, "f_{2*}"] & H_n(C_*) \arrow[r, "\partial"] \arrow[d, "f_{3*}"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(B'_*) \arrow[r] & H_{n+1}(C'_*) \arrow[r, "\partial"] & H_n(A'_*) \arrow[r] & H_n(B'_*) \arrow[r] & H_n(C'_*) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

□

Corollary 4.3.2. Let $(X, A) \to (Y, B)$ be a morphism in $\mathbf{Top_2}$. If any two of:
- $X \to Y$
- $A \to B$
- $(X, A) \to (Y, B)$

induce isomorphisms in homology, then so does the third map.

We move toward one of the main theorems in this part of the course– the *excision* theorem, which will give us the ability to "cut out" pieces of spaces and study the homology of the cut space in terms of the relative homology of the original space. This theorem was one of the first that helped me gain intuition for relative homology; although it is hard to prove, the statement of the theorem is very insightful and shows homology has plenty of interesting local properties.

Definition 4.6 (Excision)**.** A triple (X, A, U) is called **excisive** if:

$$\bar{U} \subseteq int(A) \tag{73}$$

where $\bar{\cdot}$ is the closure operator. Then the map:

$$(X \setminus U, A \setminus U) \hookrightarrow (X, A) \tag{74}$$

is called an **excision**.

We will typically use this when $X \in \mathbf{Top}$ and $U \subseteq A \subseteq X$ are two subspaces. Note that an excision is quite literally a cutting out of a space; we cut out the subspace U and then are able to embed this reduced space into the original. This should not change the relative homology of X relative to A, since the relative homology should depend on only the information of X which lies outside A, since it depends on the quotient $S_*(X)/S_*(A)$. We make this precise with the excision theorem:

Theorem 4.4 (Excision)**.** An excision is a homology isomorphism, i.e. if (X, A, U) is excisive, then:

$$H_*(X \setminus U, A \setminus U) \cong H_*(X, A) \tag{75}$$

In this way, an excision allows us to cut out a subset of A, and the relative homology will just "forget about it" and be left unchanged.

5. Products on Homology

5.1. **The cross product.** The cross product is a map $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, which we will show extends to homology and has many nice properties. The basic intuition is to separate our simplices into factors and then map them into a larger simplex in the product space in a natural way. We will prove the existence by induction. Recall Theorem 3.4, which we restate here for convenience:

Theorem 5.1. There is a natural bilinear map $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, called the **cross product**, such that the following conditions hold.

(1) $\times$ is natural, so given maps $f : X \to X'$ and $g : Y \to Y'$, the following diagram commutes:

$$\begin{array}{ccc} S_p(X) \times S_q(Y) & \xrightarrow{\times} & S_{p+q}(X \times Y) \\ \downarrow{\scriptstyle (f_*, g_*)} & & \downarrow{\scriptstyle (f \times g)_*} \\ S_p(X') \times S_q(Y') & \xrightarrow{\times} & S_{p+q}(X' \times Y') \end{array} \tag{76}$$

where f_* and g_* are the maps induced on the n-chains and $(f \times g)_*$ is the map induced by $f \times g : X \times Y \to X' \times Y', (x, y) \mapsto (f(x), g(y))$.

(2) $\times$ is bilinear, i.e. $(a + a') \times b = a \times b + a' \times b$ and $a \times (b + b') = a \times b + a \times b'$.

(3) d is an *anti-derivation* with respect to $\times$, so

$$d(a \times b) = da \times b + (-1)^p a \times db$$

(4) $\times$ is normalized. For $x \in X$, $y \in Y$, and each $a \in S_p(X), b \in S_q(Y)$, given the inclusions $j_x : Y \hookrightarrow X \times Y, y \mapsto (x, y)$ and $\iota_y : X \to X \times Y$, the induced maps $(j_x)_* : S_*(Y) \to S_*(X \times Y)$ and $(\iota_x)_* : S_*(X) \to S_*(X \times Y)$ satisfy:

$$(j_x)_*(b) = C_x^0 \times b$$

Theorem 5.2. There is a natural bilinear normalized map:

$$\times : H_q(X) \times H_q(X) \to H_{p+q}(X \times Y) \tag{77}$$

5.2. Kunneth Formulas.