8.851 Final Project: Chiral Lagrangians in Lattice QCD

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Abstract

This project will discuss how to implement the chiral Lagrangian in lattice QCD. We will begin with a brief introduction to lattice QCD and introduce Wilson fermions, before turning to a discussion of chiral symmetry on the lattice and introducing Ginsparg-Wilson fermions. We will show how to construct the chiral Lagrangian for these types of fermions by regarding the finite lattice-spacing corrections to the QCD action as external operators. A spurion analysis similar to that for the QCD mass term can be conducted to add these corrections to the chiral Lagrangian, and we shall compute these spurions and present the chiral Lagrangian for Wilson and Ginsparg-Wilson fermions in lattice QCD up to order a^2.

1 Introduction

Quantum Chromodynamics (QCD) is the theory of the strong nuclear force. It is a $SU(3)$ gauge theory coupled to 6 flavors of fermions– the quarks. At low energies, the coupling of QCD is large and perturbation theory cannot be applied to the theory, which means physics must be extracted by non-perturbative means. One such way to do this is to formulate QCD as a **lattice gauge theory** by discretizing spacetime– the advantage of this is that the path integral becomes a finite (albeit large) dimensional integral which can be evaluated on a computer via a Monte Carlo simulation.

Unfortunately due to the complexity of QCD, it is often difficult to approximate the path integral with a low degree of uncertainty. A major limitation is inverting the discretized Dirac operator $\not{D} + m_q$ as a N^2 dimensional matrix, where N is the number of lattice sites. When m is small this operator is near singular, and inversions are much noisier and more expensive. To reduce this cost and error, lattice computations are often done where the light quarks are taken to have a larger-than-physical mass, and then extrapolated to the limit of physical quark mass.

These quark mass extrapolations all rely on chiral perturbation theory (χPT), which offers us valuable insight into the structure of low-energy QCD. Looking at QCD in the massless quark limit allows us to study the spontaneous breaking of chiral symmetry, which yields the chiral Lagrangian. Using the chiral Lagrangian, we can analytically study the light spectrum of QCD, and make predictions about how changing the light quark masses will change the physics of QCD.

The chiral Lagrangian was developed for continuum studies of QCD, and as a result it does not contain any discretization artifacts which are generated by discretizing spacetime and putting it on a finite lattice. As a result, it is of interest to study how to put the chiral Lagrangian on the lattice. We will see that it is more difficult than in the continuum case because lattice actions have a plethora of terms which break chiral symmetry. However, we can deal with these terms by treating them as spurions, much like how the mass term is added to the continuum chiral Lagrangian.

We will begin with a review of chiral perturbation theory and the spurion technique. Next, we will give an overview of lattice QCD and discuss its degrees of freedom. We will then use the Symanzik action as an example and show how to add discretization artifacts from the action to the chiral Lagrangian. Finally, we will end by studying Ginsparg-Wilson fermions in chiral perturbation theory, which are a specific type of lattice fermion that behaves nicely under chiral symmetry.

2 Preliminaries

2.1 The chiral Lagrangian

The Lagrangian for massless QCD in the Minkowski continuum is

$$\mathcal{L} = \overline{\psi} i \not{D} \psi = \overline{\psi}_L i \not{D} \psi_L + \overline{\psi}_R i \not{D} \psi_R \tag{1}$$

where the left and right handed components of ψ are defined as $\psi_L = \frac{1-\gamma_5}{2}\psi$ and $\psi_R = \frac{1+\gamma_5}{2}\psi$. This decomposition makes it evident that massless QCD is invariant under the symmetries $(\psi_L, \psi_R) \mapsto (L\psi_L, R\psi_R)$, where $(L, R) \in SU(N_f)_L \times SU(N_f)_R$[1] are flavor rotations and N_f is the number of massless quarks. The chiral symmetry of $\mathcal{L}$ under $G := SU(N_f)_L \times SU(N_f)_R$ is spontaneously broken by the quark condensate $\langle q\overline{q}\rangle \neq 0$ down to the subgroup $H := SU(N_f)_V$, where $SU(N_f)_V$ is the vector subgroup of G given by $H = \{(V, V) \in G : V \in SU(N_f)\}$.

The chiral Lagrangian is a theory of the Goldstone bosons which result from the spontaneous symmetry breaking pattern $G \to H$. It can be matched to QCD at low energies, and used to make

[1] The symmetry group here could be taken to be $U(N_f)_L \times U(N_f)_R$, but the axial $U(1)$ pieces is broken by anomalies and $U(1)_V$ symmetry leads to $B - L$ conservation.

phenomenological predictions about the light degrees of freedom of QCD. To construct the chiral Lagrangian, we begin with a field Σ for the Goldstone bosons. Since Goldstone bosons live in the quotient space G/H, Σ must parameterize this quotient, and by exhibiting such a parameterization we can determine how Σ must transform under a chiral rotation. Let $(g_L, g_R)\ mod\ H$ be an arbitrary element in G/H. Then we may write:

$$(g_L, g_R)\ mod\ H = (g_L g_R^\dagger, 1)(g_R, g_R)\ mod\ H = (g_L g_R^\dagger, 1)\ mod\ H \tag{2}$$

as (g_R, g_R) lives in the vector subgroup H. This means that we can parameterize G/H with $\Sigma := g_L g_R^\dagger \in SU(N_f)$, and hence we see that Σ must transform under $(L, R) \in G$ as:

$$\Sigma \mapsto L\Sigma R^\dagger \tag{3}$$

We can expand $\Sigma(x) := \exp\left(\frac{2i\Pi^a(x)\tau^a}{f}\right)$, where τ^a are the generators of $SU(N_f)$, and the Π^a will be either a triplet or octet of mesons, depending on whether we take $N_f = 2$ or $N_f = 3$. The lowest order chiral Lagrangian can now be written down with the most general set of terms consistent with $\Sigma \mapsto L\Sigma R^\dagger$, which is:

$$\mathcal{L}_\chi = \frac{f^2}{4}\, tr\left\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right\} \tag{4}$$

This can be expanded in the meson fields Π^a to compute their dynamics and interactions.

We can add other interactions to this Lagrangian via a **spurion** analysis, which we will later generalize for the case of lattice field theory. To do this, we will take an operator not invariant under chiral symmetry and add it to the QCD Lagrangian with a Wilson coefficient which we will take to be a dynamical field instead of simply a coupling. This field is called the spurion, and taken to transform under G in such a way that the entire term is invariant under G. Once this is done, this term is added to the chiral Lagrangian and set equal to its initial value.

An explicit example of this method is done by adding the QCD mass term to $\mathcal{L}_\chi$. The mass term $m\overline{\psi}\psi = m(\overline{\psi}_L\psi_R\overline{\psi}_R\psi_L)$ couples together the right and left handed components of ψ, and thus is only invariant under the vector subgroup $SU(N_f)_V$ and not the entire group G. To make this invariant under G, we promote m to a field M. We take M to transform under $(L, R) \in G$ as $M \mapsto LMR^\dagger$, which makes $\overline{\psi}_L M\psi_R + \overline{\psi}_R M^\dagger\psi_L$ invariant under chiral symmetry. This term can now be added to $\mathcal{L}_\chi$ at lowest order:

$$\Delta\mathcal{L}_\chi = \mu\, tr\left\{M\Sigma^\dagger + M^\dagger\Sigma\right\} \tag{5}$$

where μ is a coupling. Once this term is added, we set it to its constant value $M_0 = diag(m_1, m_2, ...)$ and obtain:

$$\mathcal{L}_\chi = \frac{f^2}{4} tr\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} - \mu\, tr\left\{M_0\left(\Sigma + \Sigma^\dagger\right)\right\} \tag{6}$$

When we consider the chiral Lagrangian on the lattice, we will add in discretization artifacts to the Lagrangian as spurions. Before we can do this, we must discuss some background from lattice gauge theory.

2.2 Lattice QCD

This section will outline the background we need to study chiral Lagrangians on the lattice; specifically, we will define our theory and the operators of relevance to us. We begin by making some initial definitions: denote our spacetime lattice with spacing a by Λ, and Wick rotate to imaginary time so that Λ is a Euclidean lattice[2]. In the full theory of QCD, the dynamical fields in the path integral are the quark fields ψ and gluon fields A_μ. When we discretize QCD, we will still work with the quark fields, but instead of directly working with the gauge fields A_μ we will work with **link fields** $U_\mu(x)$ which transform in the following way under a gauge transformation $\Omega : \Lambda \to SU(3)$:

$$U_\mu(x) \xrightarrow{\Omega(x)} \Omega(x)U_\mu(x)\Omega(x+a\hat{\mu})^\dagger \tag{7}$$

Here $\hat{\mu}$ is the unit vector in the μ direction, and $x \in \Lambda$ denotes a site in the lattice. These link fields are the discrete counterpart of a Wilson line in continuum QFT, and may be expanded as $U_\mu(x) = \exp(iaA_\mu(x))$. They serve the role of a connection between fibers at different points in Λ, and hence they allow us to define a **discrete covariant derivative** ∇_μ as:

$$\nabla_\mu \psi(x) := \frac{U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)}{a} \tag{8}$$

∇_μ will take the place of $D_\mu = \partial_\mu + igA_\mu$ in our lattice theory.

Using this, we can discretize the Euclidean Lagrangian $\overline{\psi}(\not{D} + m)\psi$. Note that discretization is not unique, as different choices of lattice actions can yield the same continuum limit. The simplest choice of discretization of the Dirac operator $\not{D}$ is known as the **Wilson operator** D_W:

$$D_W := \frac{1}{2}\left\{\gamma^\mu(\nabla_\mu + \nabla_\mu^* - ar\nabla_\mu^*\nabla^\mu)\right\} \tag{9}$$

where r is a parameter which is typically set to 1 in lattice calculations. There are two parts to the Wilson operator: the first is the naive discretization of $\not{D}$ as $\frac{1}{2}\gamma^\mu(\nabla_\mu + \nabla_\mu^*)$. The second term proportional to $a\nabla_\mu^*\nabla^\mu$ is added to deal with the so-called **doubling problem**. The discrete nature of the lattice gives rise to a periodic Brillouin zone in momentum space, of period $2\pi/a$. The naive discretization of $\not{D}$ has a pole at $p^2 = 0$, but because of this periodicity also has poles at each edge of the Brillouin zone, which correspond to unphysical particles which are called doublers. The second term in D_W is added to cancel out these poles and remove the doublers, leaving only the physical pole in D_W at $p^2 = 0$. Adding this term to D_W is completely valid because it vanishes as we take the continuum limit. The Wilson operator thus gives us an action for **Wilson fermions**:

$$\mathcal{S}_W = a^4 \sum_x \overline{\psi}(D_W + m_q)\psi \tag{10}$$

Upon discretization of our theory, the Euclidean symmetry group $O(4)$ breaks down to its finite subgroup $H(4)$. If we are given a 4-vector X^μ, this reduction in symmetry means that we can form more invariants under $H(4)$ from X^μ than we originally had under $O(4)$. In the continuum case, the only Lorentz invariant is $X^2 = X_\mu X^\mu$, and any multiple or power of this quantity. However, under $H(4)$, we can form the quantities:

$$X^{[2n]} := \sum_\mu X_\mu^{2n} \tag{11}$$

which will be invariant under all hypercubic symmetries for $n \in \mathbb{Z}$. Effectively, this means that when we examine the operators that we can add to the chiral Lagrangian via a spurion analysis, we can also consider operators of this form which are *not* Lorentz invariants but *are* $H(4)$ invariants.

[2]We must Wick rotate so that the Boltzmann factor of e^{iS} in the path integral becomes a valid probability density e^{-S}, in order to perform any computations at all.

Equation 10 provides a simple lattice action which is quite useful in practice and allows for efficient computation of observables. However, because we have mutilated our original theory of QCD by putting it on a finite lattice, we also must deal with discretization artifacts. To see where these come from, let us examine a toy model of discretization in 1D. We will approximate the derivative of a function $f'(x)$ with a discrete difference operator δ on a lattice with spacing a. We may Taylor expand $f(x \pm a)$ as:

$$f(x \pm a) = f(x) \pm af'(x) + \frac{a^2}{2}f''(x) \pm \frac{a^3}{6}f^{(6)}(x) + \mathcal{O}(a^4) \tag{12}$$

The discrete difference of f thus approximates $f'(x)$ as:

$$\delta f(x) = \frac{f(x+a) - f(x-a)}{2a} = f'(x) + \frac{a^2}{6}f^{(3)}(x) + \mathcal{O}(a^3) \tag{13}$$

Note that the a^2 correction is written as the continuum operator $f^{(3)}(x)$, not a discrete one.

We see that this allows us to quantify the error we impose from discretization in powers of a. We can also think about ways to *reduce* this error that we impose by adding extra terms to the difference operator in such a way that correction terms come in at a higher order in a. For example, we can eliminate the term of order a^2 in the difference expansion in Equation 13 by adding in a term $D^{(3)}[f]$ at order a^2 in the power counting which proportional to $f^{(3)}(x)$ in the continuum limit. Here, we will take:

$$D^{(3)}[f](x) = \frac{f(x+2a) - 2f(x+a) + 2f(x-a) - f(x-2a)}{2a^3} = f^{(3)}(x) + \mathcal{O}(a^2) \tag{14}$$

It is now evident that using the modified difference operator $\Delta := \delta - \frac{a^2}{6}D^{(3)}$ will eliminate the discretization artifacts in Equation 13 at order a^2:

$$\Delta f(x) = f'(x) + \mathcal{O}(a^4) \tag{15}$$

In such a manner, we can in principle continue to remove discetization artifacts order by order.

On the lattice, this procedure is called **Symanzik improvement**, and leads to a power counting EFT formulation of corrections to the Wilson action. Modifying our difference operators amounts to adding higher dimensional continuum operators to $\mathcal{L}_W$ and tuning the Wilson coefficients to eliminate these discretization artifacts. We will thus generally expand the lattice action as a power series in a:

$$S_S = a^4 \sum_x \left(a^{-1}\mathcal{L}_{-1} + a^0\mathcal{L}_0 + a^1\mathcal{L}_1 + a^2\mathcal{L}_2 + ...\right) \tag{16}$$

where each $\mathcal{L}_k$ contains continuum operators of dimension $4 + k$. Note that $a^{-1}\mathcal{L}_{-1} + a^0\mathcal{L}_0$ will be the usual Wilson Lagrangian.

2.3 Ginsparg-Wilson fermions

Wilson fermions are cheap computationally and the simplest type of lattice fermion which are used in real computations, but unfortunately they play rather poorly with chiral symmetry. To see this, note that the key relation for chiral symmetry in the continuum is:

$$\{\gamma_5, \not{D}\} = 0 \tag{17}$$

as this allows the massless QCD Lagrangian to split into a sum $\overline{\psi}_L \not{D} \psi_L + \overline{\psi}_R \not{D} \psi_R$ to have independent pieces with definite chirality.

Unfortunately on the lattice, we do not have this property. For Wilson fermions governed by the Dirac operator in Equation 9, we see the Wilson Dirac operator D_W contains the term $a\overline{\psi}\nabla^2\psi$ which connects the different chiral components of ψ. Even in the massless limit, this term will break chiral symmetry. More generally, it was shown by Nielson and Ninomiya [3] that *any attempt* to remove the doublers from a lattice regularized theory would result in such a breaking of this anticommutation relation, and thus the essence of chiral symmetry in lattice theories must be reformulated.

Ginsparg and Wilson [4] proposed an alternative symmetry on the lattice that acts as chiral symmetry, and indeed goes into chiral symmetry in the continuum limit $a \to 0$. The modified anticommutation relation is known as the **Ginsparg-Wilson equation**:

$$\{\gamma_5, D_G\} = aD_G\gamma_5 D_G \tag{18}$$

where we will denote by D_G any lattice Dirac operator which satisfies this property. The Ginsparg-Wilson equation is not satisfied by Wilson Dirac operator D_W, but solutions of it are known to exist and can be implemented[3].

Suppose that D_G satisfies the Ginsparg-Wilson equation. Then $\overline{\psi}D_G\psi$ is invariant under a modified chiral rotation of the fields defined by:

$$\psi \mapsto \exp\left(i\epsilon\gamma_5\left(1 - \frac{1}{2}aD\right)\right)\psi \qquad \bar{\psi} \mapsto \bar{\psi}\exp\left(i\epsilon\left(1 - \frac{1}{2}aD\right)\gamma_5\right) \tag{19}$$

hence we can take the Noether current generated by this symmetry to be our new definition of a chiral current.

To make manifest the modified chiral symmetry of the Lagrangian, we follow an argument by Niedermayer [7] and introduce modified chiral projectors which act as the standard projectors $\frac{1\pm\gamma_5}{2}$ in the continuum limit:

$$\hat{\gamma}_5 := \gamma_5(1 - aD) \qquad \hat{P}_\pm := \frac{1 \pm \hat{\gamma}_5}{2} \tag{20}$$

Because D satisfies Equation 18, these new projectors satisfy $D\hat{P}_\pm = P_\mp D$, which is a very similar algebra to the standard chiral projectors. This means that if we introduce modified chiral fields:

$$\psi_L = \hat{P}_-\psi; \qquad \psi_R = \hat{P}_+\psi; \qquad \bar{\psi}_L = \bar{\psi}P_+; \qquad \bar{\psi}_R = \bar{\psi}P_- \tag{21}$$

then our Lagrangian splits into chiral components $\mathcal{L} = \bar{\psi}_L D\psi_L + \bar{\psi}_R D\psi_R$ as in the continuum case, and has chiral symmetry $SU(N_f)_L \times SU(N_f)_R$ as in the continuum case.

3 Chiral Lagrangians in Lattice QCD

3.1 Wilson fermions

We are now in a position to put the chiral Lagrangian on the lattice. We will begin by adding Wilson fermions to the chiral Lagrangian, as they are often cheaper computationally and simpler to use. We will consider discretization artifacts in the Symanzik EFT as external fields, and add them to the chiral Lagrangian as spurions. We begin by examining the operators which we may

[3]See, for example, Neuberger's **overlap operator** in Reference [5].

add to our theory. The first two terms are those making up the usual QCD Lagrangian:

$$\mathcal{O}(a^{-1}) \ : \ \overline{\psi}\psi \tag{22}$$
$$\mathcal{O}(a^{0}) \ : \ m_0\overline{\psi}\psi, \psi \not{D}\psi \tag{23}$$
$$\tag{24}$$

Note that terms which are equal modulo a mass coefficient can be taken to be the same, as we can simply absorb these extra terms into our definitions of the couplings by redefining them, i.e. above we can take $m'\overline{\psi}\psi := (a^{-1} + m_0)\overline{\psi}\psi$.

The operators that contribute at order a are dimension 5 operators which are invariant under C, P, and $H(4)$ rotations. There are two such operators:

$$\mathcal{O}_1^{(5)} = \overline{\psi} i\sigma_{\mu\nu}F^{\mu\nu}\psi \qquad \mathcal{O}_2^{(5)} = \overline{\psi}\vec{D}^2\psi + h.c. \tag{25}$$

$\mathcal{O}_2^{(5)}$ can be eliminated using the equations of motion in favor of $m^2\overline{\psi}\psi$ and $\mathcal{O}_1^{(5)}$, which can be shown to hold to all orders in perturbation theory [1]. Therefore, we need only consider $\mathcal{O}_1^{(5)}$, which we will call the **Pauli term**. This transforms like the mass term $m\overline{\psi}\psi$ under chiral transformations, and we will need a corresponding spurion field for it in $\mathcal{L}_\chi$.

At dimension 6, we must consider two types of operators: those which break chiral symmetry, and those which do not. There are six different dimension 6 operators which break chiral symmetry:

$$\begin{array}{lll} \mathcal{O}_1^{(6)} = (\overline{\psi}\psi)^2 & \mathcal{O}_2^{(6)} = (\overline{\psi}\gamma_5\psi)^2 & \mathcal{O}_3^{(6)} = (\overline{\psi}\sigma_{\mu\nu}\psi)^2 \\ \mathcal{O}_4^{(6)} = (\overline{\psi}t^a\psi)^2 & \mathcal{O}_5^{(6)} = (\overline{\psi}t^a\gamma_5\psi)^2 & \mathcal{O}_6^{(6)} = (\overline{\psi}t^a\sigma_{\mu\nu}\psi)^2 \end{array} \tag{26}$$

Each of these symmetry breaking structures must be added to $\mathcal{L}_\chi$ with a spurion field. Of the operators which preserve chiral symmetry, all but one can be incorporated into the chiral Lagrangian by a redefinition of couplings or renormalization coefficients. The exceptional operator is:

$$\mathcal{O}_7^{(6)} = \sum_\mu \overline{\psi}\gamma_\mu D_\mu D_\mu D_\mu \psi \tag{27}$$

$\mathcal{O}_7^{(6)}$ is special because it is the first operator we have seen which breaks $O(4)$ symmetry, and this symmetry breaking should be accounted for in the chiral Lagrangian at the appropriate order.

$\mathcal{L}_\chi$ will power count in the parameters p^2, m, and a. The leading order (LO) and next to leading order (NLO) parts of the Lagrangian are $\mathcal{L}_\chi = \mathcal{L}_\chi^{(0)} + \mathcal{L}_\chi^{(1)} + ...$, where these terms are:

$$\begin{aligned} &\mathcal{L}_\chi^{(0)} \in \mathcal{O}(p^2, m, a) \\ &\mathcal{L}_\chi^{(1)} \in \mathcal{O}(p^4, p^2 m, p^2 a, m^2, ma, a^2) \end{aligned} \tag{28}$$

Now we can move toward a spurion analysis of the LO and NLO chiral Lagrangians. First, we must ask: when are two spurions to be considered equal? Each symmetry breaking term will not, in fact, contribute its own spurion. Two terms which transform under chiral symmetry in the same way and have the same power counting in m and a will contribute the same spurion to the chiral Lagrangian, as we only care about the symmetry patterns of the spurion field and can always make a field redefinition to absorb another spurion which transforms identically.

We can now use this analysis to determine the general structure of the LO and NLO chiral Lagrangians. At dimension 5, the Pauli term contributes a spurion A which is first order in the lattice spacing a and transforms as:

$$A \mapsto LAR^\dagger \tag{29}$$

This transformation keeps the term $A_{ij}\overline{(}\psi_L)_i i\sigma_{\mu\nu}F^{\mu\nu}(\psi_R)_j + h.c.$ invariant under chiral transformations, and thus it can be added to the chiral Lagrangian. The A field will be set to the constant value of $A_0 = aI$, where I is the $N_f \times N_f$ identity matrix in flavor space.

The fermion bilinears of dimension 6 will receive two types of spurions, which we will denote by B and C. They transform as:

$$\begin{aligned} B &:= B_1 \otimes B_2 \mapsto (LB_1R^\dagger) \otimes (LB_2R^\dagger) \\ C &:= C_1 \otimes C_2 \mapsto (RC_1L^\dagger) \otimes (LC_2R^\dagger) \end{aligned} \tag{30}$$

This will keep each bilinear in Equation 26 invariant under chiral rotations. For example, each term comprising the operator $\mathcal{O}_1^{(6)} = (\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L)^2$ can be kept invariant by coupling to $B, C, B^\dagger$, or $C^\dagger$, as each term transforms like some variant of:

$$B_{ijkl}(\overline{\psi}_L)_i(\psi_R)_j(\overline{\psi}_L)_k(\psi_R)_l = (\overline{\psi}_L B_1 \psi_R)(\overline{\psi}_L B_2 \psi_R) \tag{31}$$

Using the transformation $\Sigma \mapsto L\Sigma R^\dagger$, we can thus write out the new terms which will appear in $\mathcal{L}_\chi$ at LO and NLO once the spurion fields are set to their constant values. These are:

$$\begin{aligned} \mathcal{O}(a) &: tr\{A\Sigma^\dagger + A^\dagger\Sigma\} \to a\, tr\{\Sigma + \Sigma^\dagger\} \\ \mathcal{O}(a^2) &: tr\{B_1\Sigma^\dagger\}tr\{B_2\Sigma^\dagger\} \to a^2 tr\{\Sigma^\dagger\}^2 \\ &: tr\{B_1\Sigma^\dagger B_2\Sigma^\dagger\} \to a^2 tr\{\Sigma^\dagger\Sigma^\dagger\} \\ &: tr\{C_1\Sigma\}\, tr\{C_2\Sigma^\dagger\} \to a^2 tr\{\Sigma\}\, tr\{\Sigma^\dagger\} \end{aligned} \tag{32}$$

Note that $\Sigma = \exp(2i\Pi/f)$ is unitary, hence $tr\{C_1\Sigma C_2\Sigma^\dagger\} \to a^2\, tr\{\Sigma\Sigma^\dagger\} = a^2 N_f$ and we need not include this combination of spurions. Furthermore, we do not consider terms like $tr\{B_1C_1\}\, tr\{B_2C_2^\dagger\}$, as these come in at order a^4 and will not appear in the LO or NLO Lagrangian.

Using these terms and the mass spurion as $\hat{m} = diag(m_1, ..., m_{N_f}$, we can expand the chiral Lagrangian to NLO. At first order, the chiral Lagrangian looks much like it previously did by adding the mass term, with an additional coupling proportional to the lattice spacing:

$$\mathcal{L}_\chi^{(0)} = \frac{f^2}{4}\left[tr\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} - B_0\, tr\left\{\hat{m}(\Sigma+\Sigma^\dagger)\right\} - a\, W_0\, tr\left\{\Sigma + \Sigma^\dagger\right\}\right] \tag{33}$$

At NLO, we get a variety of new terms that depend on the lattice spacing. We will not expand the entire NLO Lagrangian for the sake of brevity, but we will note a few pertinent factors. Here, $\mathcal{L}_\chi^{cont\,(1)}$ is the NLO chiral Lagrangian in the continuum. The entire NLO Lagrangian can be found in Equation 22 of [2].

$$\mathcal{L}_\chi^{(1)} = \mathcal{L}_\chi^{cont\,(1)} + a\, W_4\, tr\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\}\, tr\left\{\Sigma + \Sigma^\dagger\right\} + ... \tag{34}$$

$$... + a\, W_8\, tr\left\{\hat{m}(\Sigma^\dagger\Sigma^\dagger + \Sigma\Sigma)\right\} + ... \tag{35}$$

$$... + a^2 W_8'\, tr\left\{\Sigma^\dagger\Sigma^\dagger + \Sigma\Sigma\right\} \tag{36}$$

This completes our study of the chiral Lagrangian for Wilson fermions. We will next examine a different lattice action which is more suited to chiral symmetry, a mixed action comprised of Wilson fermions and Ginsparg-Wilson fermions.

3.2 Mixed action

As mentioned before, Wilson fermions wreak havoc on chiral symmetry. In many cases, they perform so poorly with chiral symmetry that chiral extrapolation is not even possible, because they quark masses are unable to enter the regime where $\mathcal{L}_\chi$ describes the dominant dynamics. Because of this, it is of interest to consider other types of actions which use fermions that are more chirally symmetric, for example Ginsparg-Wilson fermions. Ginsparg-Wilson fermions are too expensive to model a full lattice theory with, so we will pick and choose our battles: we will consider a **mixed action** with *Wilson sea quarks* and *Ginsparg-Wilson valence and ghost quarks*. This action has the advantage that it less computationally intensive than a full Ginsparg-Wilson theory, but more chiral symmetry than a theory with only Wilson fermions.

We will consider a theory with N_f Wilson sea quarks and N_V Ginsparg-Wilson valence quarks. The sea degrees of freedom will be contained in ψ_S, and the valence and ghost degrees of freedom will be contained in ψ_V. In flavor space, ψ_S is a N_f dimensional spinor, and ψ_V is a $2N_V$ dimensional spinor, as it contains valence quarks and ghosts. The action is:

$$S = a^4 \sum_x \left(\overline{\psi}_S D_W \psi_S + \overline{\psi}_V D_G \psi_V \right) \tag{37}$$

Note that since the ghost fields in ψ_V have bosonic statistics, the group of chiral symmetry transformations on ψ_V is *not* $SU(2N_V)$, but rather $G_{val} := SU(N_V|N_V)$, where $SU(N_V|N_V)$ is the graded group of superunitary transformations U which leave the $2N_V \times 2N_V$ dimensional matrix $diag(1, ..., 1, -1, ..., -1)$. We denote by G_M the full group of chiral symmetries of the mixed action, and this is:

$$G_M := SU(N_f + N_V|N_V)_L \times SU(N_f + N_V|N_V)_R \tag{38}$$

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